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DISPERSIVE EFFECTS FOR THE SCHRÖDINGER EQUATION ON A TADPOLE GRAPH

FELIX ALI MEHMETI, KAÏS AMMARI, AND SERGE NICAISE

Abstract. We consider the free Schrödinger group $e^{-it\frac{d^2}{dx^2}}$ on a tadpole graph $\mathcal{R}$. We first show that the time decay estimates $L^1(\mathcal{R}) \to L^\infty(\mathcal{R})$ is in $|t|^{-\frac{1}{2}}$ with a constant independent of the length of the circle. Our proof is based on an appropriate decomposition of the kernel of the resolvent. Further we derive a dispersive perturbation estimate, which proves that the solution on the queue of the tadpole converges uniformly, after compensation of the underlying time decay, to the solution of the Neumann half-line problem, as the circle shrinks to a point. To obtain this result, we suppose that the initial condition fulfills a high frequency cutoff.

Contents

1. Introduction

A characteristic feature of the Schrödinger equation is the loss of the localization of wave packets during evolution, the dispersion. This effect can be measured by L^∞-time decay, which implies a spreading out of the solutions, due to the time invariance of the L^2-norm. The well known fact that the free Schrödinger group in $\mathbb{R}^n$ considered as an operator family from L^1 to L^∞ decays exactly as $c \cdot t^{-n/2}$ follows easily from the explicit knowledge of the kernel of this group [10, p. 60].

In this paper we derive analogous L^∞-time decay estimates for Schrödinger equations on the tadpole graph (sometimes also called lasso graph).

Before a precise statement of our main result, let us introduce some notation which will be used throughout the rest of the paper.

Let $R_i, i = 1, 2$, be two disjoint sets identified with a closed path of measure equal to $L > 0$ for R_2 and to $(0, +\infty)$, for R_1, see figure 1. We set $\mathcal{R} := \cup_{k=1}^2 \overline{R}_k$. We denote by $f = (f_k)_{k=1,2} = (f_1, f_2)$ the functions on $\mathcal{R}$ taking their values in $\mathbb{C}$ and let f_k be the restriction of f to R_k.

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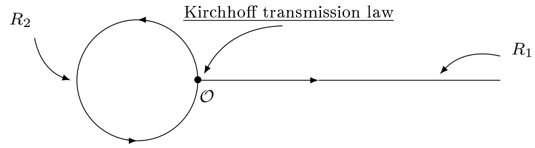

FIGURE 1. Tadpole graph

Define the Hilbert space $\mathcal{H} = \prod_{k=1}^{2} L^2(R_k) = L^2(\mathcal{R})$ with inner product

$$((u_k),(v_k))_{\mathcal{H}} = \sum_{k=1}^{2}(u_k, v_k)_{L^2(R_k)}$$

and introduce the following transmission conditions (see [9, 5]):

$$(u_k)_{k=1,2} \in \prod_{k=1}^{2} C(\overline{R_k}) \text{ satisfies } u_1(0) = u_2(0) = u_2(L), \tag{1.1}$$

$$(u_k)_{k=1,2} \in \prod_{k=1}^{2} C^1(\overline{R_k}) \text{ satisfies } \sum_{k=1}^{2} \frac{du_k}{dx}(0^+) - \frac{du_2}{dx}(L^-) = 0. \tag{1.2}$$

Let $H : \mathcal{D}(H) \subset \mathcal{H} \to \mathcal{H}$ be the linear operator on $\mathcal{H}$ defined by :

$$\mathcal{D}(H) = \left\{ (u_k) \in \prod_{k=1}^{2} H^2(R_k);\ (u_k)_{k=1,2} \text{ satisfies } (1.1), (1.2) \right\},$$

$$H(u_k) = (H_k u_k)_{k=1,2} = \left(-\frac{d^2 u_k}{dx^2} \right)_{k=1,2} = -\Delta_{\mathcal{R}}(u_k).$$

This operator H is self-adjoint and its spectrum $\sigma(H)$ is equal to $[0, +\infty)$. The self-adjointness and non-negativity of H can be shown by Friedrichs extension (see [3] for example), the fact that the spectrum is equal to the positive half-axis follows from Theorem 2.5 below.

Here, we prove that the free Schrödinger group on the tadpole graph $\mathcal{R}$ satisfies the standard $L^1 - L^\infty$ dispersive estimate. More precisely, we will prove the following theorem.

1.1. **Theorem.** *For all $t \neq 0$,*

$$\left\| e^{itH} P_{ac} \right\|_{L^1(\mathcal{R}) \to L^\infty(\mathcal{R})} \le C\, |t|^{-1/2}, \tag{1.3}$$

where C is a positive constant independent of L and t, $P_{ac}f$ is the projection onto the absolutely continuous spectral subspace and $L^1(\mathcal{R}) = \prod_{k=1}^{2} L^1(R_k)$, $L^\infty(\mathcal{R}) = \prod_{k=1}^{2} L^\infty(R_k)$.

This means that the time decay is the same as the case of a line [10, 7], a half-line [11] or star shaped networks [1, 8, 5]. Note that the proof of this result is based on an appropriate decomposition of the kernel of the resolvent that in particular gives a full characterization of the spectrum, made only of the point spectrum and of the absolutely continuous one; showing the absence of a singular continuous part. An important point is that this estimate is independent of the length L of the circle, which also follows from the fact that the problem is scale invariant, as it is shown in Remark 3.1.

Let H_0 be the negative laplacian on the half line with Neumann boundary conditions. Then holds the following dispersive perturbation estimate:

1.2. **Theorem.** *Let $0 \le a < b < \infty$. Let $u_0 \in \mathcal{H} \cap L^1(R_1)$ such that*

$$\text{supp } u_0 \subset R_1 \ . \tag{1.4}$$

Then for all $t \neq 0$, we have

$$\begin{aligned} \| e^{itH}\mathbb{I}_{(a,b)}(H)u_0 \quad &- \quad e^{itH_0}\mathbb{I}_{(a,b)}(H_0)u_0 \, \|_{L^\infty(R_1)} \\ &\le \quad t^{-1/2}L \; 2\sqrt{2}\, \Big(4(2\sqrt{b}-\sqrt{a}) + L(b-a)\Big) \; \|u_0\|_{L^1(R_1)} \, . \end{aligned}$$

This last result implies that the solution of the Schrödinger equation on the queue R_1 of the tadpole with an upper frequency cutoff tends uniformly to the solution of the half-line Neumann problem with the same upper frequency cutoff, if the initial condition has its support in the queue, after compensation of the underlying $t^{-1/2}$-decay. In physical terms the frequency cutoff makes that the localization of the signals is limited and thus they have increasing difficulties to enter into the head of the tadpole.

Without the high frequency cutoff, this result would not be possible, as the problem is scale invariant.

The paper is organized as follows. The kernel of the resolvent needed for the proof of Theorem 1.1, is given in section 2. Further all eigenfunctions of the free Hamiltonian on the tadpole are constructed. They correspond to the confined modes on the head of the tadpole, which do not interact with the queue. The interaction is described by the absolutely continuous spectrum. Technically the main point is a decomposition of the kernel of the resolvent into a meromorphic term in the whole complex plane with poles on the positive real axis and a term which is continuous on the real line but discontinuous when crossing it.

The poles are shown to be the eigenvalues of the operator, the continuous term creates the absolutely continuous spectrum. The absence of further terms proves the absence of a singular continuous spectrum. Note that in [6] the eigenvalues and eigenvectors are given without this detailed analysis of the rest of the spectrum.

In section 3 we give the proof of the main result of the paper (Theorem 1.1). To this end we replace the formula of the resolvent into Stone's formula and prepare all terms for the application of the Lemma of van der Corput.

In section 4 we prove the dispersive perturbation estimate. This is based on the comparison of the kernel of the resolvent of the tadpole operator on $R_1 \times R_2$ with the kernel of the Schrödinger operator with Neumann conditions on the half-line.

2. **Kernel of the resolvent**

Given $z \in \mathbb{C}_+ := \{z \in \mathbb{C} : \Im z > 0\}$ and $g \in L^2(\mathcal{R})$, we are looking for $u \in \mathcal{D}(H)$ solution of

$$-\Delta_{\mathcal{R}} u - z^2 u = g \text{ in } \mathcal{R}.$$

Let us use the notation $\omega = -iz$.

Hence we look for u in the form

$$\begin{aligned} u_1(x) = \int_0^{+\infty} \frac{g_1(y)}{2\omega}\Big(e^{-\omega|x-y|} - F_1(\omega)e^{-\omega(y+x)}\Big)\, dy \\ - \int_0^L \frac{g_2(y)}{2\omega}\Big(F_2(\omega)e^{-\omega(y+x)} + F_3(\omega)e^{\omega(y-x)}\Big)\, dy, \end{aligned} \tag{2.5}$$

$$\begin{aligned} u_2(x) = \int_0^{+\infty} \frac{g_1(y)}{2\omega}\Big(G_1(\omega)e^{-\omega(y+x)} + H_1(\omega)e^{-\omega(y-x)}\Big)\, dy \\ + \int_0^L \frac{g_2(y)}{2\omega}\Big(e^{-\omega|x-y|} + G_2(\omega)e^{-\omega(y+x)} + G_3(\omega)e^{\omega(y-x)} \end{aligned}$$

$$+ H_2(\omega)e^{\omega(x-y)} + H_3(\omega)e^{\omega(x+y)}\Big) \, dy, \tag{2.6}$$

where $F_i(\omega)$, $G_i(\omega)$ and $H_i(\omega)$, $i = 1, 2, 3$ are constants fixed in order to satisfy (1.1), (1.2). Indeed from these expansion, we clearly see that for $k = 1$ or 2:

$$-u_k'' + \omega^2 u_k = g_k \text{ in } R_k.$$

Now we see that the continuity condition (1.1) is satisfied if and only if

$$\begin{array}{l} G_1 + H_1 = G_1 e^{-\omega L} + H_1 e^{\omega L} = 1 - F_1, \\ 1 + G_2 + H_2 = G_2 e^{-\omega L} + H_2 e^{\omega L} = -F_2, \\ G_3 + H_3 = (1 + G_3)e^{-\omega L} + H_3 e^{\omega L} = -F_3, \end{array}$$

while Kirchhoff condition (1.2) holds if and only if

$$\begin{array}{l} F_1 + G_1(e^{-\omega L} - 1) + H_1(1 - e^{\omega L}) = -1, \\ F_2 + G_2(e^{-\omega L} - 1) + H_2(1 - e^{\omega L}) = -1, \\ F_3 + G_3(e^{-\omega L} - 1) + H_3(1 - e^{\omega L}) = e^{-\omega L}. \end{array}$$

These equations correspond to three linear systems in F_i, G_i, H_i, $i = 1, 2, 3$, whose associated matrix has a determinant $D(\omega)$ given by

$$D(\omega) = e^{\omega L}(e^{-\omega L} - 1)(e^{-\omega L} - 3).$$

Since this determinant is different from zero (as $\Im z > 0$), we deduce the following expressions:

$$F_1(\omega) = 1 + \frac{2(e^{-\omega L} + 1)}{e^{-\omega L} - 3}, \tag{2.7}$$

$$G_1(\omega) = -\frac{2}{e^{-\omega L} - 3}, \tag{2.8}$$

$$H_1(\omega) = -\frac{2e^{-\omega L}}{e^{-\omega L} - 3}, \tag{2.9}$$

$$F_2(\omega) = \frac{2}{e^{-\omega L} - 3}, \tag{2.10}$$

$$G_2(\omega) = -\frac{e^{\omega L}}{D(\omega)}, \tag{2.11}$$

$$H_2(\omega) = \frac{2 - e^{-\omega L}}{D(\omega)}, \tag{2.12}$$

$$F_3(\omega) = -\frac{2e^{-2\omega L}}{e^{-\omega L} - 3}, \tag{2.13}$$

$$G_3(\omega) = \frac{e^{-\omega L}}{D(\omega)}, \tag{2.14}$$

$$H_3(\omega) = \frac{e^{-\omega L}}{D(\omega)}(2e^{-\omega L} - 3). \tag{2.15}$$

Inserting these expressions in (2.5)-(2.6), we have obtained the next result.

2.1. **Theorem.** *Let $f \in \mathcal{H}$. Then, for $x \in \mathcal{R}$ and $z \in \mathbb{C}$ such that $\Im z > 0$, we have*

$$[R(z^2, H)f](x) = \int_{\mathcal{R}} K(x, x', z^2) f(x') \, dx', \tag{2.16}$$

where the kernel K is defined as follows:

$$
\begin{aligned}
(2.17) K(x,y,z^2) &= \frac{1}{2iz}\left(e^{iz|x-y|} - F_1(-iz)e^{iz(x+y)}\right), \forall x,y \in R_1,\\
(2.18) K(x,y,z^2) &= -\frac{1}{2iz}\left(F_2(-iz)e^{iz(y+x)} + F_3(-iz)e^{-iz(y-x)}\right), \forall x \in R_1, y \in R_2,\\
(2.19) K(x,y,z^2) &= \frac{1}{2iz}\Big(e^{iz|x-y|} + G_2(-iz)e^{iz(y+x)} + G_3(-iz)e^{-iz(y-x)}\\
&+ H_2(-iz)e^{-iz(x-y)} + H_3(-iz)e^{-iz(x+y)}\Big), \forall x,y \in R_2,\\
(2.20) K(x,y,z^2) &= \frac{1}{2iz}\left(G_1(-iz)e^{iz(y+x)} + H_1(-iz)e^{iz(y-x)}\right), \forall x \in R_2, y \in R_1.
\end{aligned}
$$

As usual, to obtain the resolution of the identity of H, we want to use the limiting absorption principle that consists to pass to the limit in $K(x,y,z^2)$ as $\Im z$ goes to zero. But in view of the presence of the factor $e^{izL}-1$ in the denominator of G_2, G_3, H_2, H_3, this limit is a priori not allowed. This factor comes from the circle R_2 and suggests that the point spectrum is distributed in the whole continuous spectrum. This is indeed the case has the next results will show.

2.2. **Lemma.** *For all $k \in \mathbb{N}^*$, the number $\lambda_{2k}^2 = \frac{4k^2\pi^2}{L^2}$ is an eigenvalue of H of the associated eigenvector $\varphi^{(2k)} \in \mathcal{D}(H)$ given by*

$$\varphi_1^{(2k)} = 0 \text{ in } R_1, \tag{2.21}$$

$$\varphi_2^{(2k)}(x) = \frac{\sqrt{2}}{\sqrt{L}}\sin(\lambda_{2k}x), \forall x \in R_2. \tag{2.22}$$

Furthermore H has no other eigenvalues.

Proof. The proof of the first assertion is direct since we readily check that $\varphi^{(2k)}$ defined by (2.21)-(2.22) is indeed in $\mathcal{D}(H)$ and satisfies $H\varphi^{(2k)} = \lambda_{2k}^2\varphi^{(2k)}$.

For the second assertion, we simply remark that if φ is an eigenvector of H of eigenvalue λ^2, then for $\lambda > 0$, we have

$$\varphi_1(x) = c_1\sin(\lambda x) + c_2\cos(\lambda x), \forall x \in R_1,$$

with $c_i \in \mathbb{C}$. But the requirement that φ_1 belongs to $L^2(R_1)$ directly implies that $c_1 = c_2 = 0$. Hence φ has to be in the form of the first assertion. In the case $\lambda = 0$, φ_1 has to be zero and therefore

$$\varphi_2(x) = c_1 + c_2x, \forall x \in R_2,$$

with $c_i \in \mathbb{C}$. By the continuity property at 0, we get $c_1 = c_1 + c_2L = 0$, hence $c_1 = c_2 = 0$. $\square$

2.3. **Remark.** *We shall see below that the eigenvalues $(\lambda_{2k}^2)_{k\in\mathbb{N}^*}$ are embedded in the continuous spectrum with corresponding eigenfunctions $\varphi^{(2k)}$ which are confined in the ring.*

At this stage we define the projection P_{pp} on the closed subspace spanned by the $\varphi^{(2k)}$'s, namely for any $f \in \mathcal{H}$, we set

$$P_{pp}f = \sum_{k=0}^{+\infty}(f, \varphi^{(2k)})_{\mathcal{H}}\varphi^{(2k)}.$$

Note that $P_{pp}f$ is different from f on R_2 because $L^2(R_2)$ is spanned by the set of eigenvectors of the Laplace operator with Dirichlet boundary conditions at 0 and L, that are the set $\{\varphi_2^{(\ell)}\}_{\ell\in\mathbb{N}^*}$, where

$$\varphi_2^{(\ell)}(x) = \frac{\sqrt{2}}{\sqrt{L}}\sin(\lambda_\ell x), \forall x \in R_2,$$

and $\lambda_\ell = \frac{\ell\pi}{L}$. Hence

$$f - P_{pp} f = \sum_{k=0}^{+\infty} \left(\int_0^L f(x) \varphi_2^{(2k+1)}(x)\, dx \right) \varphi_2^{(2k+1)}.$$

To show that our operator has no singular continuous spectrum, we shall split up the kernel $K(x, y, z^2)$ into a meromorphic part K_p in the whole complex plane with poles at the points λ_{2k} and a part K_c which is continuous on the real line but discontinuous when crossing it.

2.4. **Theorem.** *For all $z \in \mathbb{C}$ such that $\Im z > 0$, and all $x, y \in \mathcal{R}$, the kernel $K(x, y, z^2)$ defined in Theorem 2.1 admits the decomposition*

$$K(x, y, z^2) = K_c(x, y, z^2) + K_p(x, y, z^2), \tag{2.23}$$

where for $x, y \in R_2$ and $X = e^{izL}$ we have

$$\begin{aligned} K_c(x, y, z^2) &= -\frac{1}{2iz} \left(\frac{X+1}{X-3} e^{iz(x-y)} + \frac{2X(X-1)}{X-3} e^{-iz(x+y)} - 2i \frac{X-1}{X-3} \sin(zy) e^{-izx} \right) \\ &\quad -\frac{1}{2iz} \left(1 + \frac{2}{X-3} \right) \sin(zx) \sin(zy). \end{aligned}$$

and

$$K_p(x, y, z^2) = \frac{\cos\left(\frac{zL}{2}\right)}{2z \sin\left(\frac{zL}{2}\right)} \sin(zx) \sin(zy)$$

The function $z \mapsto K_c(x, y, z^2)$ is continuous on $\Im z \geq 0$ except at $z = 0$, while $z \mapsto K_p(x, y, z^2)$ is meromorphic in $\mathbb{C}$ with poles at the points λ_{2k}, $k \in \mathbb{N}^*$ and at $z = 0$.

For $x \notin R_2$ or $y \notin R_2$ we have $K_p(x, y, z^2) = 0$ and $K_c(x, y, z^2) = K(x, y, z^2)$ as defined in Theorem 2.1.

Proof. The problem only appears for x and y in R_2, since in the other cases, K has no poles and therefore in that cases we simply take $K_p = 0$. Hence we need to perform this splitting for $x, y \in R_2$.

First we transform $G_2(-iz)e^{izy} + G_3(-iz)e^{-izy}$ appropriately in order to bring out its meromorphic part that comes from the zeroes of the factor $e^{izL} - 1$ that precisely correspond to the points $z = \lambda_{2k}$. If we write for shortness $X = e^{izL}$, we see that

$$\begin{aligned} G_2(-iz)e^{izy} + G_3(-iz)e^{-izy} &= \frac{1}{(X-1)(X-3)} (X^2 e^{-izy} - e^{izy}) \\ &= \frac{1}{(X-1)(X-3)} \left((X^2 - 1) e^{-izy} + e^{-izy} - e^{izy} \right) \\ &= \frac{X+1}{X-3} e^{-izy} - \frac{2i}{(X-1)(X-3)} \sin(zy). \end{aligned}$$

In the same manner, we can show that

$$H_2(-iz)e^{izy} + H_3(-iz)e^{-izy} = \frac{2X(X-1)}{X-3} e^{-izy} - 2i \frac{X-1}{X-3} \sin(zy) + \frac{2i}{(X-1)(X-3)} \sin(zy).$$

These two expressions lead to

$$\begin{aligned} &G_2(-iz)e^{iz(y+x)} + G_3(-iz)e^{-iz(y-x)} + H_2(-iz)e^{-iz(x-y)} + H_3(-iz)e^{-iz(x+y)} \\ &= (G_2(-iz)e^{izy} + G_3(-iz)e^{-izy})e^{izx} + (H_2(-iz)e^{izy} + H_3(-iz)e^{-izy})e^{-izx} \\ &= \frac{4}{(X-1)(X-3)} \sin(zx) \sin(zy) + K_1(x, y, z), \end{aligned} \tag{2.24}$$

where

$$K_1(x, y, z^2) = \frac{X+1}{X-3} e^{iz(x-y)} + \frac{2X(X-1)}{X-3} e^{-iz(x+y)} - 2i \frac{X-1}{X-3} \sin(zy) e^{-izx}, \tag{2.25}$$

is clearly continuous up to $\Im z = 0$. We are therefore reduced to transform the factor $\frac{1}{(X-1)(X-3)}$. First as its poles correspond to the case $X = 1$, we can replace the factor $X-3$ by -2, indeed we have by partial fraction decomposition

$$\frac{1}{(X-1)(X-3)} = \frac{1}{2(X-3)} - \frac{1}{2(X-1)}.$$

Recalling that $X = e^{izL}$, we have

$$\begin{aligned}\frac{1}{X-1} = \frac{1}{e^{izL}-1} &= \frac{e^{-\frac{izL}{2}}}{e^{\frac{izL}{2}} - e^{-\frac{izL}{2}}}\\ &= -\frac{1}{2} + \frac{\cos\left(\frac{zL}{2}\right)}{2i\sin\left(\frac{zL}{2}\right)}.\end{aligned}$$

Using these two identities into (2.24), we get

$$\begin{aligned}&G_2(-iz)e^{iz(y+x)} + G_3(-iz)e^{-iz(y-x)} + H_2(-iz)e^{-iz(x-y)} + H_3(-iz)e^{-iz(x+y)}\\ &= i\frac{\cos\left(\frac{zL}{2}\right)}{\sin\left(\frac{zL}{2}\right)}\sin(zx)\sin(zy) + K_2(x,y,z),\end{aligned} \tag{2.26}$$

where

$$K_2(x,y,z^2) = K_1(x,y,z^2) + \left(1 + \frac{2}{X-3}\right)\sin(zx)\sin(zy), \tag{2.27}$$

is clearly continuous up to $\Im z = 0$.

Plugging this splitting into (2.19), we find that

$$K(x,y,z^2) = \frac{\cos\left(\frac{zL}{2}\right)}{2z\sin\left(\frac{zL}{2}\right)}\sin(zx)\sin(zy) + K_c(x,y,z^2), \forall x,y \in R_2,$$

where $K_c(x,y,z^2) = \frac{1}{2iz}K_2(x,y,z)$, is clearly continuous up to $\Im z = 0$ except at $z = 0$. This proves (2.23) with

$$K_p(x,y,z^2) = \frac{\cos\left(\frac{zL}{2}\right)}{2z\sin\left(\frac{zL}{2}\right)}\sin(zx)\sin(zy), \tag{2.28}$$

which is holomorphic on $\mathbb{C}$ with poles at $z = 0$ and $z = \lambda_{2k}$ as stated. □

With these notations, we are able to give the expression of the resolution of the identity E of H.

2.5. **Theorem.** *Take $f, g \in \mathcal{H}$ with a compact support and let $0 < a < b < +\infty$. Then for any holomorphic function h on the complex plane, we have*

$$\begin{aligned}(h(H)E(a,b)f,g)_{\mathcal{H}} &= -\frac{1}{\pi}\int_{\mathcal{R}}\Big(\int_{(a,b)} h(\lambda)\int_{\mathcal{R}} f(x')\Im K_c(x,x',\lambda)\,dx'\,d\lambda\Big)\bar{g}(x)dx\\ &+ \sum_{k\in\mathbb{N}^*:a<\lambda_{2k}^2<b} h(\lambda_{2k}^2)(f,\varphi^{(2k)})_{\mathcal{H}}(\varphi^{(2k)},g)_{\mathcal{H}},\end{aligned} \tag{2.29}$$

where for all $\lambda > 0$, $K_c(x,x',\lambda)$ is defined in Lemma 2.4.

Proof. First by Stone's formula, we have (see for instance Lemma 3.13 of [2] or Proposition 4.5 of [4])

$$(h(H)E(a,b)f,g)_{\mathcal{H}} = \lim_{\delta\to 0}\lim_{\varepsilon\to 0}\frac{1}{2i\pi}\left(\int_{a+\delta}^{b-\delta}[h(\lambda)R(\lambda - i\varepsilon, H) - R(\lambda + i\varepsilon, H)]\,d\lambda f, g\right)_{\mathcal{H}}.$$

First using Lemma 2.4, we can write

$$(h(H)E(a,b)f,g)_{\mathcal{H}} = I_c + I_p,$$

where

$$I_c = \lim_{\delta\to 0}\lim_{\varepsilon\to 0}\frac{1}{2i\pi}\left(\int_{a+\delta}^{b-\delta} h(\lambda)\left[R_c(\lambda-i\varepsilon,H)-R_c(\lambda+i\varepsilon,H)\right]d\lambda f,g\right)_{\mathcal{H}}$$

$$I_p = \lim_{\delta\to 0}\lim_{\varepsilon\to 0}\frac{1}{2i\pi}\left(\int_{a+\delta}^{b-\delta} h(\lambda)\left[R_p(\lambda-i\varepsilon,H)-R_p(\lambda+i\varepsilon,H)\right]d\lambda f,g\right)_{\mathcal{H}}.$$

where R_p (resp. R_c) is the operator corresponding to the kernel K_p (resp. K_c).

As $R_c(\lambda-i\varepsilon,H)=\overline{R_c(\lambda+i\varepsilon,H)}$, we can write

$$\begin{aligned}
I_c &= -\frac{1}{\pi}\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\left(\int_{a+\delta}^{b-\delta} h(\lambda)\Im R_c(\lambda+i\varepsilon,H)d\lambda f,g\right)_{\mathcal{H}}\\
&= -\frac{1}{\pi}\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\int_{a+\delta}^{b-\delta} h(\lambda)(\Im R_c(\lambda+i\varepsilon,H)f,g)_{\mathcal{H}}d\lambda\\
&= -\frac{1}{\pi}\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\int_{a+\delta}^{b-\delta} h(\lambda)\int_{\mathcal{R}}\int_{\mathcal{R}}\Im K_c(x,x',\lambda+i\varepsilon)f(x')dx'g(x)dxd\lambda.
\end{aligned}$$

At this stage, we take advantage of Theorem 2.1 and Lemma 2.4. First by (2.17) to (2.20) and by (2.23), we see that

$$K_c(x,y,\lambda+i\varepsilon)\longrightarrow K_c(x,y,\lambda),\ \text{as } \varepsilon\longrightarrow 0.$$

Furthermore we see that

$$|K_c(x,y,\lambda+i\varepsilon)|\leq\frac{C}{|\lambda+i\varepsilon|},\forall\lambda\in(a,b),x,y\in\mathcal{R},$$

for $x,y\in\mathcal{R}$ for some positive constant C independent of x,y. This allows to pass to the limit in $\varepsilon\longrightarrow 0$ by using the convergence dominated theorem to obtain that

$$\begin{aligned}
&\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\int_{a+\delta}^{b-\delta} h(\lambda)\int_{\mathcal{R}}\int_{\mathcal{R}}\Im K_c(x,x',\lambda+i\varepsilon)f(x')dx'g(x)dxd\lambda\\
&\qquad=\int_a^b h(\lambda)\int_{\mathcal{R}}\int_{\mathcal{R}}\Im K_c(x,x',\lambda)f(x')dx'g(x)dxd\lambda.
\end{aligned}$$

Hence it remains to manage the term I_p. As before we can write

$$I_p=\frac{1}{2i\pi}\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\int_{a+\delta}^{b-\delta}\int_{R_2}\int_{R_2} h(\lambda)(K_p(x,x',\lambda-i\varepsilon)-K_p(x,x',\lambda+i\varepsilon))f(x')dx'g(x)dxd\lambda.$$

First by Lemma 2.7 below, we can show that

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{a+\delta}^{b-\delta}\int_{R_2}\int_{R_2} h(\lambda)(K_p(x,x',\lambda-i\varepsilon)-K_p(x,x',\lambda+i\varepsilon))f(x')dx'g(x)dxd\lambda\\
&=\lim_{\varepsilon\to 0}\int_{a+\delta}^{b-\delta}\int_{R_2}\int_{R_2}(h(\lambda-i\varepsilon)K_p(x,x',\lambda-i\varepsilon)-h(\lambda+i\varepsilon)K_p(x,x',\lambda+i\varepsilon))f(x')dx'g(x)dxd\lambda.
\end{aligned}$$

Now, for a fixed $\delta>0$, we can always assume that $a+\delta$ and $b-\delta$ are always different from λ_{2k} and therefore

$$\lim_{\varepsilon\to 0}\int_{-\varepsilon}^{\varepsilon}\int_{R_2}\int_{R_2} h(c_\delta+iy)K_p(x,x',c_\delta+iy)f(x')dx'g(x)dxdy=0,$$

for $c_\delta=a+\delta$ or $b-\delta$. Consequently if we define the contour $C_{\varepsilon,\delta}$ by the lines $\lambda-i\varepsilon$, $\lambda+i\varepsilon$ with $\lambda\in(a+\delta,b-\delta)$ and $a+\delta+iy$, $b-\delta+iy$, with $y\in(-\varepsilon,\varepsilon)$, we deduce that

$$I_p=-\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\frac{1}{2i\pi}\int_{C_{\varepsilon,\delta}}\int_{R_2}\int_{R_2} h(\lambda)K_p(x,x',\lambda)f(x')dx'g(x)dxd\lambda.$$

Now reminding the expression (2.28), we perform the change of variable $\mu = \sqrt{\lambda}$, that leads to

$$
\begin{aligned}
&-\frac{1}{2i\pi}\int_{C_{\varepsilon,\delta}}\int_{R_2}\int_{R_2} h(\lambda)K_p(x,x',\lambda)f(x')dx'g(x)dxd\lambda \\
&= \frac{1}{2i\pi}\int_{D_{\varepsilon,\delta}} 2\mu\ h(\mu^2)\int_{R_2}\int_{R_2}\frac{\cos\left(\frac{\mu L}{2}\right)}{\sin\left(\frac{\mu L}{2}\right)}\sin(\mu x)\sin(\mu x')f(x')dx'g(x)dxd\mu,
\end{aligned}
$$

where $D_{\varepsilon,\delta} = \{\sqrt{\lambda} : \lambda \in C_{\varepsilon,\delta}\}$ is another contour in the complex plane. At this stage we can apply residue theorem and deduce, after simple calculations of the residues, that

$$
\begin{aligned}
&-\frac{1}{2i\pi}\int_{C_{\varepsilon,\delta}}\int_{R_2}\int_{R_2} h(\lambda)K_p(x,x',\lambda)f(x')dx'g(x)dxd\lambda \\
&= \frac{2}{L}\sum_{k\in\mathbb{N}^*:a+\delta<\lambda_{2k}^2<b-\delta} h(\lambda_{2k}^2)\left(\int_{R_2} f(x')\sin(\lambda_{2k}x')dx'\right)\left(\int_{R_2} g(x)\sin(\lambda_{2k}x)dx\right) \\
&= \sum_{k\in\mathbb{N}^*:a+\delta<\lambda_{2k}^2<b-\delta} h(\lambda_{2k}^2)(f,\varphi^{(2k)})_{\mathcal{H}}(\varphi^{(2k)},g)_{\mathcal{H}}.
\end{aligned}
$$

This proves the result by passing to the limit in $\delta \to 0$. □

2.6. **Corollary.** *The operator H has no singular spectrum and*

$$P_{ac}f = f - P_{pp}f, \forall f \in \mathcal{H}.$$

2.7. **Lemma.** *Under the assumption of the previous theorem, one has*

$$
\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{a+\delta}^{b-\delta}\int_{R_2}\int_{R_2} h(\lambda)K_p(x,x',\lambda\pm i\varepsilon)f(x')dx'g(x)dxd\lambda \\
&= \lim_{\varepsilon\to 0}\int_{a+\delta}^{b-\delta}\int_{R_2}\int_{R_2} h(\lambda\pm i\varepsilon)K_p(x,x',\lambda\pm i\varepsilon)f(x')dx'g(x)dxd\lambda.
\end{aligned}
$$

Proof. The proof is based on the use of the Lebesgue's convergence theorem. Let us prove it in the case $\lambda + i\varepsilon$. Writing

$$
\begin{aligned}
\int_{a+\delta}^{b-\delta} h(\lambda)K_p(x,x',\lambda+i\varepsilon)d\lambda &= \int_{a+\delta}^{b-\delta}(h(\lambda)-h(\lambda+i\varepsilon))K_p(x,x',\lambda+i\varepsilon)d\lambda \\
&+ \int_{a+\delta}^{b-\delta} h(\lambda+i\varepsilon)K_p(x,x',\lambda+i\varepsilon)d\lambda,
\end{aligned}
$$

we only need to show that

$$\lim_{\varepsilon\to 0}\int_{a+\delta}^{b-\delta}\int_{R_2}\int_{R_2}(h(\lambda)-h(\lambda+i\varepsilon))K_p(x,x',\lambda+i\varepsilon)f(x')dx'g(x)dxd\lambda = 0.$$

Since for any $\lambda \neq \lambda_{2k}^2$, one has

$$(h(\lambda)-h(\lambda+i\varepsilon))K_p(x,x',\lambda+i\varepsilon) \to 0, \text{ as } \varepsilon \to 0,$$

to apply Lebesgue's convergence theorem, it suffices, for instance, to show that $(h(\lambda)-h(\lambda+i\varepsilon))K_p(x,x',\lambda+i\varepsilon)$ is uniformly bounded (in ε). But in view of the definition (2.28) of K_p, we only need to estimate the ratio

$$q(\lambda,\varepsilon) := \frac{h(\lambda)-h(\lambda+i\varepsilon)}{\sin\left(\frac{\sqrt{\lambda+i\varepsilon}L}{2}\right)}.$$

Now using a Taylor expansion, we can say that for z small enough, say $|z| < \eta$, we have

$$\sin\left(\frac{\sqrt{\lambda_{2k}^2+z}L}{2}\right) = \cos(k\pi)\frac{zL}{2\lambda_{2k}} + o(z).$$

Hence applying this property to $\lambda - \lambda_{2k}^2 + i\varepsilon$ for one $k \in \mathbb{N}^*$, we find that
$$\sin\left(\frac{\sqrt{\lambda + i\varepsilon L}}{2}\right) \sim \cos(k\pi)\frac{L}{2\lambda_{2k}}(\lambda - \lambda_{2k}^2 + i\varepsilon),$$
for $(\lambda - \lambda_{2k}^2)^2 + \varepsilon^2 < \eta^2$. Hence for $|\lambda - \lambda_{2k}^2| < \frac{\eta}{2}$ and $\varepsilon < \frac{\eta}{2}$, we deduce that
$$\left|\sin\left(\frac{\sqrt{\lambda + i\varepsilon)}L}{2}\right)\right| \sim \frac{L}{2\lambda_{2k}}|\lambda - \lambda_{2k}^2 + i\varepsilon| \geq \frac{L}{2\lambda_{2k}}\varepsilon.$$
Since h is holomorphic, we clearly have
$$|h(\lambda) - h(\lambda + i\varepsilon)| \leq C\varepsilon, \forall \lambda \in [a, b],$$
for some $C > 0$ (independent of λ and ε). Therefore for any λ such that $|\lambda - \lambda_{2k}^2| < \frac{\eta}{2}$ for some $k \in \mathbb{N}^*$, and for any $\varepsilon < \frac{\eta}{2}$, one has
$$|q(\lambda, \varepsilon)| \leq C',$$
for some $C' > 0$ (independent of λ and ε).

It then remains to treat the case where $|\lambda - \lambda_{2k}^2| \geq \frac{\eta}{2}$ for all $k \in \mathbb{N}^*$. But in that case, as
$$\sqrt{\lambda + i\varepsilon} - \lambda_{2k} = \frac{\lambda + i\varepsilon - \lambda_{2k}^2}{\sqrt{\lambda + i\varepsilon} - \lambda_{2k}},$$
for ε small enough, we get
$$|\sqrt{\lambda + i\varepsilon} - \lambda_{2k}| \geq C\text{"}\eta,$$
for some $C\text{"} > 0$ (independent of λ and ε). This implies that
$$\left|\sin\left(\frac{\sqrt{\lambda + i\varepsilon)}L}{2}\right)\right| \geq \delta,$$
for some $\delta > 0$ (independent of λ and ε) and again implies that $q(\lambda, \varepsilon)$ is uniformly bounded in that case as well.

□

3. **Proof of the main result**

For any $0 < a < b < \infty$, by Theorem 2.5 for any $x, y \in \mathcal{R}$, we have found the following expression for the kernel of the operator $e^{itH}\mathbb{I}_{(a,b)}P_{ac}$:
$$\int_0^{+\infty} e^{it\lambda}\mathbb{I}_{(a,b)}(\lambda)E_{ac}(d\lambda)(x, y) = -\frac{1}{\pi}\int_a^b e^{it\lambda}\mathbb{I}_{(a,b)}(\lambda)\Im K_c(x, y, \lambda)d\lambda,$$
and by the change of variables $\lambda = \mu^2$, we get
$$\int_0^{+\infty} e^{it\lambda}\mathbb{I}_{(a,b)}E_{ac}(d\lambda)(x, y) = -\frac{2}{\pi}\int_{\sqrt{a}}^{\sqrt{b}} e^{it\mu^2}\mathbb{I}_{(a,b)}(\mu^2)\Im K_c(x, y, \mu^2)\mu\, d\mu.$$
Now recalling the definition of K_c, we have to distinguish between the following cases:

(1) If $x, y \in R_1$, then
$$2i\mu K_c(x, y, \mu^2) = e^{i\mu|x-y|} - F_1(-i\mu)e^{i\mu(x+y)}.$$
Hence in that case, we have to estimate
$$\left|\int_{\sqrt{a}}^{\sqrt{b}} e^{it\mu^2}e^{i\mu|x-y|}d\mu\right|,$$
and
$$\left|\int_{\sqrt{a}}^{\sqrt{b}} e^{it\mu^2}F_1(-i\mu)e^{i\mu(x+y)}d\mu\right|.$$

The first term is directly estimated by van der Corput Lemma [12, p. 332]:

$$\left|\int_{\sqrt{a}}^{\sqrt{b}} e^{it\mu^2} e^{i\mu|x-y|} d\mu\right| \leq \frac{4\sqrt{2}}{\sqrt{t}}, \forall\, t > 0.$$

For the second term by using (2.7), we have

$$\begin{aligned}\left|\int_{\sqrt{a}}^{\sqrt{b}} e^{it\mu^2} F_1(-i\mu) e^{i\mu(x+y)} d\mu\right| \quad &\leq \quad \left|\int_{\sqrt{a}}^{\sqrt{b}} e^{it\mu^2} e^{i\mu(x+y)} d\mu\right| \\ &+ \quad 2\left|\int_{\sqrt{a}}^{\sqrt{b}} e^{it\mu^2} e^{i\mu(x+y)} \frac{e^{i\mu L}+1}{e^{i\mu L}-3} d\mu\right|.\end{aligned}$$

Again the first term of this right-hand side is estimated by van der Corput Lemma, while for the second one we use the Neumann series

$$\frac{1}{e^{i\mu L}-3} = -\frac{1}{3(1-\frac{e^{i\mu L}}{3})} = -\frac{1}{3}\sum_{k=0}^{+\infty} \frac{e^{ik\mu L}}{3^k},$$

to obtain (owing to Fubini's theorem)

$$\left|\int_{\sqrt{a}}^{\sqrt{b}} e^{it\mu^2} e^{i\mu(x+y)} \frac{e^{i\mu L}+1}{e^{i\mu L}-3} d\mu\right| \leq \frac{1}{3}\sum_{k=0}^{+\infty}\frac{1}{3^k}\left|\int_{\sqrt{a}}^{\sqrt{b}} e^{it\mu^2} e^{i\mu(x+y)+ik\mu}(e^{i\mu L}+1) d\mu\right|.$$

Again applying van der Corput Lemma at each term we obtain

$$\left|\int_{\sqrt{a}}^{\sqrt{b}} e^{it\mu^2} e^{i\mu(x+y)} \frac{e^{i\mu L}+1}{e^{i\mu L}-3} d\mu\right| \leq \frac{8\sqrt{2}}{3\sqrt{t}}\sum_{k=0}^{+\infty}\frac{1}{3^k} = \frac{4\sqrt{2}}{\sqrt{t}}.$$

All together we have proved that

$$\left|\int_0^{+\infty} e^{it\lambda} \mathbb{I}_{(a,b)}(\lambda) E_{ac}(d\lambda)(x,y)\right| \leq \frac{9\sqrt{2}}{\sqrt{t}}, \forall x, y \in R_1, t > 0. \tag{3.30}$$

(2) If $x \in R_2$ and $y \in R_1$, then

$$2i\mu K_c(x,y,\mu^2) = G_1(-i\mu)e^{i\mu(y+x)} + H_1(-i\mu)e^{i\mu(y-x)}$$

and in view of the form of G_1 and H_1, the same arguments as before imply that

$$\left|\int_0^{+\infty} e^{it\lambda} \mathbb{I}_{(a,b)}(\lambda) E_{ac}(d\lambda)(x,y)\right| \leq \frac{C}{\sqrt{t}}, \forall x \in R_2, y \in R_1, t > 0, \tag{3.31}$$

where $C > 0$ is independent of x, y, t, a, b and L.

(3) If $x \in R_1$ and $y \in R_2$, then

$$2i\mu K_c(x,y,\mu^2) = -(F_2(-i\mu)e^{i\mu(y+x)} + F_3(-i\mu)e^{-i\mu(y-x)})$$

and from the form of F_2 and F_3, we deduce as before that

$$\left|\int_0^{+\infty} e^{it\lambda} \mathbb{I}_{(a,b)}(\lambda) E_{ac}(d\lambda)(x,y)\right| \leq \frac{C}{\sqrt{t}}, \forall x \in R_1, y \in R_2, t > 0, \tag{3.32}$$

where $C > 0$ is independent of x, y, t, a, b and L.

(4) If $x, y \in R_2$, then owing to Lemma 2.4, we have

$$2i\mu K_c(x,y,\mu^2) = K_2(x,y,\mu),$$

with K_2 defined by (2.27). But again the form of K_2 (and of K_1) allows to deduce as before that

$$\left|\int_0^{+\infty} e^{it\lambda} \mathbb{I}_{(a,b)}(\lambda) E_{ac}(d\lambda)(x,y)\right| \leq \frac{C}{\sqrt{t}}, \forall x, y \in R_2, t > 0, \tag{3.33}$$

where $C > 0$ is independent of x, y, t, a, b and L.

The estimates (3.30) to (3.33) imply that for all $f \in L^2(\mathcal{R}) \cap L^1(\mathcal{R})$

$$\|e^{itH}\mathbb{I}_{(a,b)}(H)P_{ac}f\|_\infty \leq \frac{2C}{\sqrt{t}}\|f\|_1, \forall t > 0, \tag{3.34}$$

where $C > 0$ is independent of t, a, b and L.

As $e^{itH}\mathbb{I}_{(a,b)}(H)P_{ac}f$ converges to $e^{itH}P_{ac}f$ in $L^2(\mathcal{R})$ as $a \to 0$ and $b \to \infty$, by extracting a subsequence, we have that

$$e^{itH}\mathbb{I}_{(a,b)}(H)P_{ac}f \to e^{itH}P_{ac}f \text{ a.e.},$$

and therefore (3.34) implies that

$$\|e^{itH}P_{ac}f\|_\infty \leq \frac{2C}{\sqrt{t}}\|f\|_1, \forall t > 0, \tag{3.35}$$

where $C > 0$ is independent of t and L. By density we conclude the proof of Theorem 1.1.

3.1. **Remark.** The above proof underlines that the constant C appearing in the $L^1 - L^\infty$ estimate (1.3) is independent of the length L of R_2. But this independence can be proved with the help of the following scaling argument. Let u be a solution of the Schrödinger equation on the tadpole graph $\mathcal{R}$ with initial datum u_0, i.e., solution of

$$\begin{aligned}
\frac{du_k}{dt} + i\frac{d^2u_k}{dx^2} = 0 \text{ in } R_k \times \mathbb{R}, k = 1, 2,\\
u_1(0,t) = u_2(0^+,t) = u_2(L^-,t), \text{ in } \mathbb{R},\\
\sum_{k=1}^{2}\frac{du_k}{dx}(0^+,t) - \frac{du_2}{dx}(L^-,t) = 0, \text{ in } \mathbb{R},\\
u(\cdot,0) = u_0, \text{ in } \mathcal{R}.
\end{aligned}$$

Then we perform the change of variables $x = L\hat{x}$, $t = L^2\hat{t}$ that transform $\mathcal{R} \times \mathbb{R}$ into $\hat{\mathcal{R}} \times \mathbb{R}$, where $\hat{\mathcal{R}} = \hat{R}_2 \cup \hat{R}_1$, $\hat{R}_1 = (0, +\infty)$ and $\hat{R}_2$ is a closed path of length 1. Hence by setting $\hat{u}_0(\hat{x}) = u_0(x)$ and $\hat{u}(\hat{x},\hat{t}) = u(x,t)$, we see that $\widehat{P_{ac}u_0} = \hat{P}_{ac}\hat{u}_0$ and that $\hat{u}$ is solution of the Schrödinger equation on the tadpole graph $\hat{\mathcal{R}}$ with initial datum $\hat{u}_0$. Hence applying the estimate (1.3) on $\hat{\mathcal{R}}$, we find that

$$\|e^{i\hat{t}\hat{H}}\hat{P}_{ac}\hat{u}_0\|_{L^\infty(\hat{\mathcal{R}})} \leq \frac{\hat{C}}{\sqrt{\hat{t}}}\|\hat{u}_0\|_{L^1(\hat{\mathcal{R}})}, \forall \hat{t} > 0,$$

where $\hat{C}$ is a positive constant independent of t. As $\|e^{itH}P_{ac}u_0\|_{L^\infty(\mathcal{R})} = \|e^{i\hat{t}\hat{H}}\hat{P}_{ac}\hat{u}_0\|_{L^\infty(\hat{\mathcal{R}})}$ and $\|u_0\|_{L^1(\mathcal{R})} = L\|\hat{u}_0\|_{L^1(\hat{\mathcal{R}})}$, we find that, for all $t > 0$

$$\begin{aligned}
\|e^{itH}P_{ac}u_0\|_{L^\infty(\mathcal{R})} &= \|e^{i\hat{t}\hat{H}}\hat{P}_{ac}\hat{u}_0\|_{L^\infty(\hat{\mathcal{R}})}\\
&\leq \frac{\hat{C}}{\sqrt{\frac{t}{L^2}}}\|\hat{u}_0\|_{L^1(\hat{\mathcal{R}})}\\
&\leq \frac{\hat{C}}{\sqrt{t}}\|u_0\|_{L^1(\mathcal{R})}.
\end{aligned}$$

This proves that (1.3) holds on $\mathcal{R}$ with $C \leq \hat{C}$. Since the converse implication also holds, we have shown that $C = \hat{C}$ in (1.3). Therefore if (1.3) holds for a certain C and L_0, then it holds for all L with the same C.

4. The shrinking circle limit for initial conditions in frequency bands

In this section we consider the limit of the solution of the Schrödinger equation on the tadpole as the circumference of the circle tends to zero. To obtain a result, we need the crucial hypothesis, that the initial condition has an upper cutoff frequency. We shall use the formulas for the kernel $K(x, y, \lambda)$ of the resolvent $(H-\lambda)^{-1}$ of the negative laplacian H on the tadpole and the kernel $K_0(x, y, \lambda)$ of the resolvent $(H_0-\lambda)^{-1}$ of the negative laplacian H_0 on the half line with Neumann boundary conditions: let us recall that by equation (2.17) we have

$$K(x, y, z^2) = \frac{1}{2iz}\left(e^{iz|x-y|} - 1 + \frac{2(e^{izL}+1)}{e^{izL}-3}e^{iz(x+y)}\right),$$

for $\Im z > 0$ and $x, y \in R_1 \cong (0, \infty)$. Further we have

$$K_0(x, y, z^2) = \frac{1}{2iz}\left(e^{iz|x-y|} + e^{iz(x+y)}\right),$$

for $\Im z > 0$ and $x, y \geq 0$, which can be checked by direct calculations. Inserting this expression in Stone's formula and applying the limiting absorption principle yields

$$\begin{aligned}\left(e^{itH_0}\mathbb{I}_{(a,b)}(H_0)u_0\right)(x) &= \frac{2}{\pi}\int_{\sqrt{a}}^{\sqrt{b}} e^{it\mu^2}\cos(\mu x)\Big(\int_0^\infty \cos(\mu y)u_0(y)\ dy\Big)d\mu \\ &= \mathcal{C}^{-1}\Big[\mathbb{I}_{(\sqrt{a},\sqrt{b})}(\mu)e^{it\mu^2}(\mathcal{C}u_0)(\mu)\Big](x) =: u(t, x)\end{aligned}$$

where $\mathcal{C}$ and $\mathcal{C}^{-1}$ are the cosine transform and its inverse. This last representation makes it easy to check, that for $-\infty < a < b < \infty$ the function u is smooth and satisfies

$$\left\{\begin{array}{ll} iu_t - u_{xx} = 0, & t, x \geq 0 \\ u_x(t, 0) = 0, & t \geq 0 \\ u(0, \cdot) = u_0. & \end{array}\right. \tag{4.36}$$

Now we calculate the difference of the kernels of the resolvents of the tadpole problem on its queue and of the half line problem with the Neumann boundary condition:

4.1. **Proposition.** *For $x, y \in R_1 \cong (0, \infty)$ and $\mu > 0$ we have*

$$K(x, y, \mu^2) - K_0(x, y, \mu^2) = -\frac{2(e^{izL}+1)}{e^{izL}-3}e^{iz(x+y)}. \tag{4.37}$$

Proof. Let $x, y \in R_1 \cong (0, \infty)$, $\mu > 0$. Then

$$\begin{aligned} & K(x, y, \mu^2) - K_0(x, y, \mu^2) \\ = \ & \frac{1}{2i\mu}\left(e^{i\mu|x-y|} - \left(1 + \frac{2(e^{i\mu L}+1)}{e^{i\mu L}-3}\right)\ e^{i\mu(x+y)}\right) - \frac{1}{2i\mu}\left(e^{i\mu|x-y|} + e^{i\mu(x+y)}\right) \\ = \ & \frac{1}{2i\mu}\left(-\left(1 + \frac{2(e^{i\mu L}+1)}{e^{i\mu L}-3}\right) - 1\right)e^{i\mu(x+y)} \\ = \ & -\frac{1}{\mu}\,\frac{2(e^{i\mu L}-1)}{e^{i\mu L}-3}\,e^{i\mu(x+y)}. \end{aligned}$$

□

By a simple substitution, we derive from E. Stein [12], p. 334 the following variant of the Lemma of van der Corput for $k = 2$:

4.2. **Proposition.** *Suppose that $\Phi : (a,b) \to \mathbb{R}$ is smooth and satisfies $\mid \Phi''(x) \mid \geq M > 0$ for $x \in (a,b), \lambda > 0$ and that $\Psi \in W^{1,1}(a,b)$. Then*

$$\left| \int_a^b e^{i\lambda\Phi(x)} \Psi(x) dx \right| \leq \frac{8}{(\lambda M)^{1/2}} \left(\mid \Psi(b) \mid + \int_a^b \mid \Psi'(x) \mid dx \right). \tag{4.38}$$

Now we are able to compare the Schrödinger time-evolution on the queue of the tadpole and on the half line.

4.3. **Theorem.** *Let $\left(e^{itH}\mathbb{I}_{(a,b)}(H)P_{ac}\right)(x,y)$ and $\left(e^{itH_0}\mathbb{I}_{(a,b)}(H_0)\right)(x,y)$ be the kernels of the operator groups in the brackets. For $0 \leq a < b < \infty$, $t \neq 0$ and $x, y \in R_1 \cong (0,\infty)$ we have*

$$\begin{aligned} \left(e^{itH}\mathbb{I}_{(a,b)}(H)P_{ac}\right)(x,y) \quad &- \quad \left(e^{itH_0}\mathbb{I}_{(a,b)}(H_0)\right)(x,y) \\ &= \quad \int_{\sqrt{a}}^{\sqrt{b}} e^{i(t\mu^2+\mu(x+y))} \frac{4(1-e^{i\mu L})}{e^{i\mu L}-3} e^{i\mu(x+y)} d\mu, \end{aligned} \tag{4.39}$$

and

$$\begin{aligned} \Big| \left(e^{itH}\mathbb{I}_{(a,b)}(H)P_{ac}\right)(x,y) \quad &- \quad \left(e^{itH_0}\mathbb{I}_{(a,b)}(H_0)\right)(x,y) \Big| \\ &\leq \quad t^{-1/2} L \; 2\sqrt{2} \, \Big(4(2\sqrt{b} - \sqrt{a}) + L(b-a) \Big). \end{aligned} \tag{4.40}$$

Proof. Ad (4.39): Thanks to proposition 4.1 we have

$$\begin{aligned} &\left(e^{itH}\mathbb{I}_{(a,b)}(H)P_{ac}\right)(x,y) - \left(e^{itH_0}\mathbb{I}_{(a,b)}(H_0)\right)(x,y) \\ = \; &\int_a^b e^{it\lambda} \Big(K(x,y,\lambda) - K_0(x,y,\lambda) \Big) \, d\lambda \\ = \; &\int_{\sqrt{a}}^{\sqrt{b}} e^{it\mu^2} \Big(K(x,y,\mu^2) - K_0(x,y,\mu^2) \Big) 2\mu \; d\mu \\ = \; &\int_{\sqrt{a}}^{\sqrt{b}} e^{i(t\mu^2+\mu(x+y))} \frac{4(1-e^{i\mu L})}{e^{i\mu L}-3} d\mu. \end{aligned}$$

Ad (4.40): In view of applying proposition 4.2, we put

$$\Phi(\mu) = \mu^2 + \mu \frac{x+y}{t} \quad \text{and} \quad \psi(\mu) = \frac{4(1-e^{i\mu L})}{e^{i\mu L}-3}.$$

Then $\Phi''(\mu) = 2$ and thus

$$\begin{aligned} \Big| \left(e^{itH}\mathbb{I}_{(a,b)}(H)P_{ac}\right)(x,y) \quad &- \quad \left(e^{itH_0}\mathbb{I}_{(a,b)}(H_0)\right)(x,y) \Big| \\ &= \quad \left| \int_{\sqrt{a}}^{\sqrt{b}} e^{it\mu} \Psi(\mu) dx \right| \\ &\leq \quad \frac{8}{(2t)^{1/2}} \left(\mid \Psi(\sqrt{b}) \mid + \int_{\sqrt{a}}^{\sqrt{b}} \mid \Psi'(\mu) \mid d\mu \right). \end{aligned} \tag{4.41}$$

We estimate the expressions on the right hand side by using

$$\mid 1 - e^{i\mu L} \mid \leq \mu L \text{ and } \mid e^{i\mu L} - 3 \mid \geq 2$$

which yields

$$\mid \Psi(\sqrt{b}) \mid = \left| \frac{4(1-e^{i\sqrt{b}L})}{e^{i\sqrt{b}L}-3} \right| \leq \frac{4\sqrt{b}L}{2} = 2\sqrt{b}L.$$

Further we have

$$\Psi'(\mu) = \frac{-4Lie^{i\mu L}}{e^{i\mu L}-3} - \frac{-4Li(1-e^{i\mu L})}{(e^{i\mu L}-3)^2}.$$

Therefore

$$| \Psi'(\mu) | \leq 2L + \mu L^2 \text{ and thus } \int_{\sqrt{a}}^{\sqrt{b}} | \Psi'(\mu) | \, d\mu \leq 2L(\sqrt{b} - \sqrt{a}) + \frac{L^2}{2}(b-a).$$

Together with (4.41) this yields the assertion. □

4.4. **Corollary.** *Let $0 \leq a < b < \infty$. Let $u_0 \in \mathcal{H} \cap L^1(R_1)$ such that*

$$\text{supp } u_0 \subset R_1 \ . \tag{4.42}$$

Then for all $t \neq 0$, we have

$$\begin{aligned} \| e^{itH} \mathbb{I}_{(a,b)}(H) u_0 \quad &- \quad e^{itH_0} \mathbb{I}_{(a,b)}(H_0) u_0 \|_{L^\infty(R_1)} \\ &\leq \quad t^{-1/2} L \ 2\sqrt{2} \ \Big(4(2\sqrt{b} - \sqrt{a}) + L(b-a) \Big) \ \|u_0\|_{L^1(R_1)} \,. \end{aligned}$$

Proof. For $x \in R_1$, condition (4.42) implies the second equality of

$$\begin{aligned} & (e^{itH} \mathbb{I}_{(a,b)}(H) \, u_0 - e^{itH_0} \mathbb{I}_{(a,b)}(H_0) \, u_0)(x) \\ = & \int_{\mathcal{R}} \left(e^{itH} \mathbb{I}_{(a,b)}(H) \right) (x,y) \, u_0(y) \ dy - \int_0^\infty \left(e^{itH_0} \mathbb{I}_{(a,b)}(H_0) \right) (x,y) \, u_0(y) \ dy \\ = & \int_0^\infty \Big[\left(e^{itH} \mathbb{I}_{(a,b)}(H) \right) (x,y) \, u_0(y) - \left(e^{itH_0} \mathbb{I}_{(a,b)}(H_0) \right) (x,y) \, u_0(y) \Big] \ dy \\ = & \int_0^\infty \Big[\left(e^{itH} \mathbb{I}_{(a,b)}(H) P_{ac} \right) (x,y) - \left(e^{itH_0} \mathbb{I}_{(a,b)}(H_0) \right) (x,y) \Big] u_0(y) \ dy. \end{aligned}$$

The support condition (4.42) implies $P_{pp} u_0 = 0$ and thus $u_0 = P_{ac} \, u_0$, which justifies the last equality. Then the assertion follows from the hypothesis $u_0 \in \mathcal{H} \cap L^1(R_1)$ and Theorem 4.3 (4.40). □

In Remark 3.1 we proved that the tadpole problem is scale invariant. In particular we showed, that if the dispersive estimate in Theorem 1.1 holds with a constant C for a given circumference L of the head of the tadpole, then it holds for arbitrary L with the same constant C.

Corollary 4.4 of Theorem 4.3 implies that the solution of the Schrödinger equation on the queue of the tadpole with an upper frequency cutoff tends uniformly to the solution of the half-line Neumann problem with the same upper frequency cutoff, if the support of the initial condition has its support in the queue, after compensation of the underlying $t^{-1/2}$-decay.

The upper frequency cutoff introduces in physical terms an upper limit for the (group) velocity of wave packets and thus a lower limit for the localization of wave packets (by an intuitive application of the uncertainty principle). Thus wave packets are large with respect to the head of the tadpole, if L is small. Therefore the upper cutoff frequency destroys the scale invariance and it becomes plausible, that the solutions of the tadpole problem tend to the solutions of the half-line Neumann problem, if the head of the tadpole shrinks to a point.

Technically this can be seen in inequality (4.41) and the subsequent arguments: we used

$$| 1 - e^{i\mu L} | \leq \mu L$$

and the inequality of Stein (Proposition 4.2), which introduced the dependence of the cutoff frequency (by μ) and the perturbation aspect ($L \to 0$). By using the triangular inequality which gives $| \ 1 - e^{i\mu L} \ | \leq 2$ and using the (pure) van der Corput estimate we would have avoided the dependence on the upper cutoff frequency, but at the same time also the perturbation aspect.

There exists also an interpretation of formula (4.39) from Theorem 4.3 : by using the expansion

$$\frac{1}{e^{i\mu L}-3} = -\frac{1}{3}\sum_{k=0}^{+\infty}\frac{e^{ik\mu L}}{3^k},$$

in the right hand side as in section 3, we obtain a series representation of the difference of the solutions of the tadpole problem on its queue and the half-line Neumann problem. The terms correspond to signals passing from the head of the tadpole into its queue after k cycles around the head.

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