Control of Partial Differential Equations: Theoretical Aspects

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1. Introduction

Controllability for partial differential equations has been extensively studied in the last twenty years and there is a vast literature on the subject treating a number of different models. Here we only present the case of linear controlled partial differential equation and we focus essentially on the wave equation (with some comments on Schrödinger equation) and on heat type equations. We do not treat the case of Korteweg de Vries equation or Stokes equation, neither the important case of nonlinear equations such as nonlinear parabolic equations or Navier-Stokes equations. For a reader who would be interested in modern developments on these subjects, we refer to the very important book by Jean-Michel Coron [6] or to some published articles (see [9], [23], [12], [14], [7] for example).

We restrict ourselves to the classical methods introduced essentially in [18] for the Hilbert Uniqueness Method (HUM) and [11] for global Carleman estimates. Other methods could have been considered, for example the method of moments (see [25]) or methods based on microlocal analysis (see [1] for example).

The present notes require some basic knowledge on the existence theory for the equations considered and on classical functional spaces (like Sobolev spaces) and functional analysis, but they should be accessible to most graduate students.

2. Introduction to Controllability

We consider an abstract linear evolution controlled system on an interval of time $(0,T)$, with $T > 0$.

$$\frac{du}{dt} + Au = Bh \quad \text{on } (0,T), \tag{2.0.1}$$

$$u(0) = u_0, \tag{2.0.2}$$

where A is an operator in the space variable, t is the time variable, B is the control operator, h is the control.

We have the choice of the control h (in a suitable space, say X) and this control acts on the system via operator B which can be unbounded.

We assume that there is a good existence theory, for example: $h \in X$ and $u_0 \in H$ give $u \in C([0,T],H)$.

What can we obtain as values of the solution at time T?

Controllability is the study of the reachable states $\{u(T)\}$ and there are several more precise notions of controllability.

2.1. *Approximate Controllability*

Approximate controllability means:

Given any $u_0 \in H$ and any $u_1 \in H$; for every $\epsilon > 0$, does there exist $h_\epsilon \in X$ such that

$$u(0) = u_0 \quad \text{and} \quad ||u(T) - u_1||_H \leq \epsilon?$$

Approximate controllability has been extensively studied in the early 1990's but the interest has decreased....

We will not develop this notion here.

2.2. *Null Controllability*

Null controllability in time T means:

Given any $u_0 \in H$, does there exist $h \in X$ such that

$$u(0) = u_0 \quad \text{and} \quad u(T) = 0?$$

2.3. *Exact Controllability*

Exact controllability in time T means (Figure 2.1):

Given any $u_0 \in H$ and $u_1 \in H$, does there exist $h \in X$ such that

$$u(0) = u_0 \quad \text{and} \quad u(T) = u_1?$$

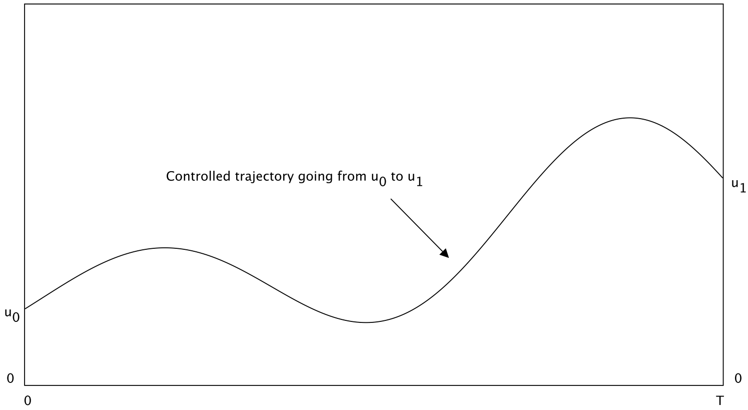

Figure 2.1 Exact controllability

This notion will be relevant for reversible systems like wave equations, Schrödinger equations, etc.

For linear **reversible** systems exact controllability is in fact equivalent to null controllability (Figure 2.2): take first $u_1 = 0$ with u_0 given which gives a first control h_0, then $u_0 = 0$ with u_1 given (for the reverse system) which gives a second control h_1 then adding up we see that the two notions are equivalent. This is no longer valid for nonlinear systems.

nonlinear systems.

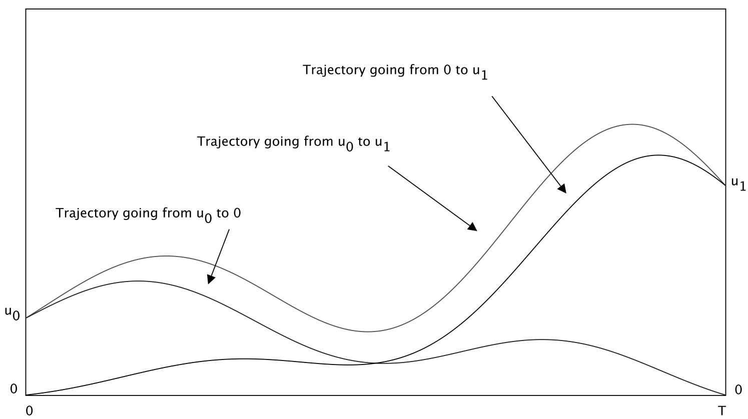

Figure 2.2 Exact controllability for reversible systems and null controllability

2.4. *Exact Controllability to Trajectories*

This notion is relevant for nonlinear systems of the form

$$\frac{du}{dt} + Au + N(u) = Bh \quad \text{in } (0, T), \tag{2.4.3}$$

$$u(0) = u_0. \tag{2.4.4}$$

We consider an "ideal" uncontrolled trajectory $\bar{u}$ solution (Figure 2.3) of

$$\frac{d\bar{u}}{dt} + A\bar{u} + N(\bar{u}) = 0 \quad \text{in } (0, T), \tag{2.4.5}$$

$$\bar{u}(0) = \bar{u}_0. \tag{2.4.6}$$

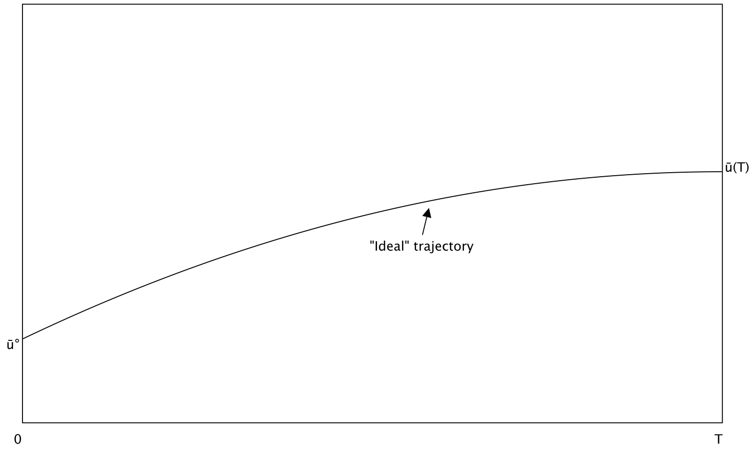

Figure 2.3 Ideal trajectory

Exact controllability to trajectories means (Figure 2.4):

Given $u_0 \in H$ can we find $h \in X$ such that at time T we have

$$u(T) = \bar{u}(T)?$$

This notion will be important for non reversible systems. For linear systems, it is equivalent (by simple difference) to null controllability.

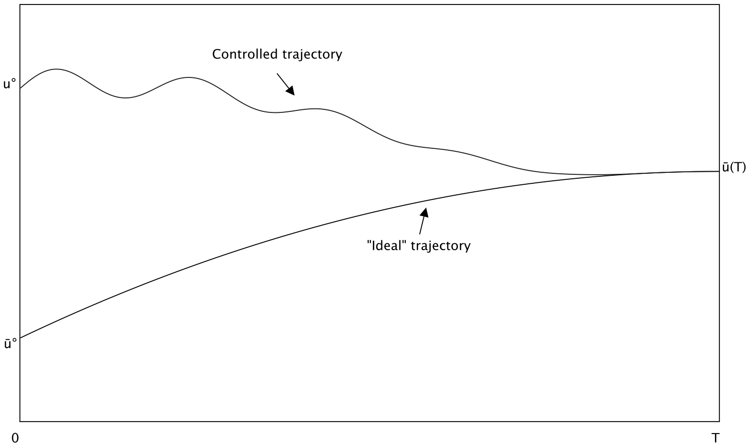

Figure 2.4 Global Exact Controllability to trajectories

We can also define the local version of exact controllability to trajectories (Figure 2.5).

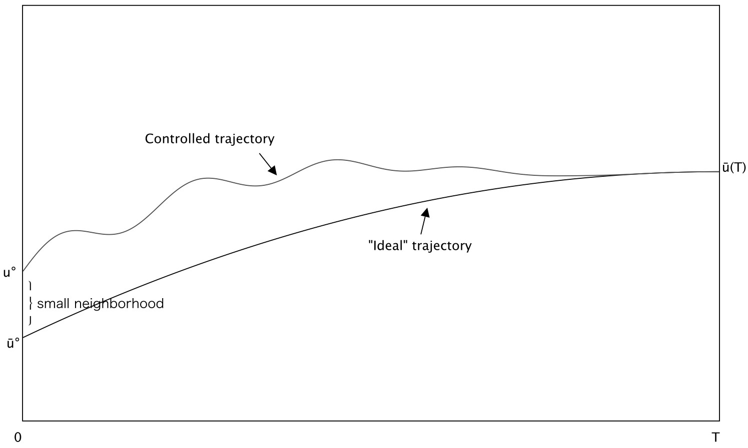

Figure 2.5 Local Exact Controllability to trajectories

3. Simple Examples

3.1. *Transport Equation in 1-D*

We consider the linear transport equation on an interval $(0, 1)$ where the control acts on the left boundary. We want to study the exact controllability for this equation.

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0 \quad \text{on } (0,1) \times (0,T), \tag{3.1.1}$$

$$u(0,t) = h(t), \quad t \in (0,T), \tag{3.1.2}$$

$$u(x,0) = u_0(x), \quad x \in (0,1). \tag{3.1.3}$$

Then the solution is given by

$$u(x,t) = \xi(x-t)$$

with

$$\xi(x) = u_0(x) \quad \text{and} \quad \xi(-t) = h(t).$$

• When $T < 1$: for $T < x < 1$ we have $u(x,T) = u_0(x-T)$ so that it is impossible to reach any u_1 on this interval.

• When $T \geq 1$: for $x \in (0,1)$, we have $u(x,T) = h(T-x)$ and we can choose $h(t) = u_1(T-t)$ for $t \in (T-1,T)$ to obtain the exact controllability result.

This simple example shows that, due to the finite speed propagation in the transport equation (Figure 3.1), we need the time T to be large enough in order to obtain exact controllability.

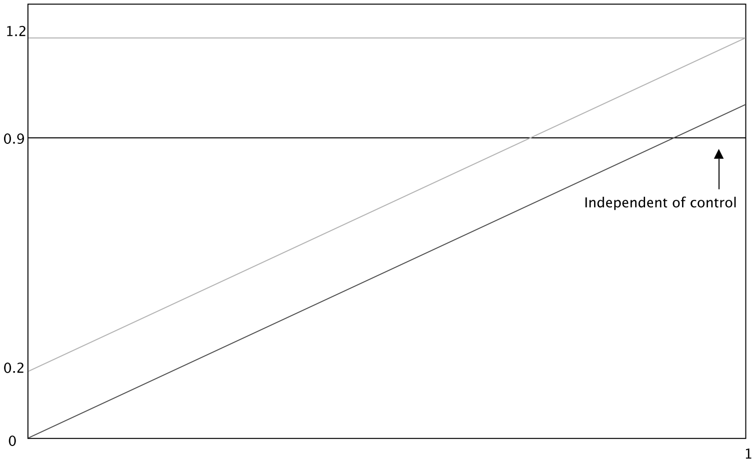

Figure 3.1 Transport equation

3.2. *Wave Equation in 1-D*

We want to study here the case of a wave equation on an interval $(0, 1)$ with 0 initial velocity (in order to simplify) and with control acting on the left boundary.

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0 \quad \text{on } (0,1)\times(0,T), \tag{3.2.4}$$

$$u(0,t) = h(t), \quad u(1,t) = 0, \quad t \in (0,T), \tag{3.2.5}$$

$$u(x,0) = u_0(x), \quad \frac{\partial u}{\partial t}(x,0) = 0, \quad x \in (0,1). \tag{3.2.6}$$

This equation can be written as a first order (in time) system

$$\frac{dY}{dt} + AY + BH = 0$$

by setting

$$Y = \begin{pmatrix} u \\ \dfrac{\partial u}{\partial t} \end{pmatrix}, \quad H = \begin{pmatrix} 0 \\ h \end{pmatrix},$$

and

$$AY = \begin{pmatrix} -u \\ -\dfrac{\partial^2 u}{\partial x^2} \end{pmatrix}.$$

Operator B is here unbounded and corresponds to boundary conditions. As the wave equation is reversible we are only interested in the null controllability which means here that we want to achieve

$$u(x,T) = 0 \quad \text{and} \quad \frac{\partial u}{\partial t}(x,T) = 0.$$

A simple calculation shows that

$$u(x,t) = \frac{1}{2}u_0(x-t) + \frac{1}{2}u_0(x+t)$$

where u_0 has been extended so that

$$u_0(t) + u_0(-t) = 2h(t) \quad \text{and} \quad u_0(1-t) + u_0(1+t) = 0.$$

- For $0 < T < 1$ and $T < x < 1$ we have, as $x - T < 1$ and $x + T < 2$

$$u(x,T) = \frac{1}{2}u_0(x-T) - \frac{1}{2}u_0(2-(x+T))$$

which is independent of h so that exact controllability is impossible.

- For $1 \leq T \leq 2$ and x small enough, we have

$$u(x,T) = h(T-x) - \frac{1}{2}u_0(2+x-T) + \frac{1}{2}u_0(2-x-t),$$

and

$$\frac{\partial u}{\partial t}(x,T) = h'(T-x) + \frac{1}{2}u_0'(2+x-T) - \frac{1}{2}u_0'(2-x-T).$$

In order to find h such that $u(x,T) = 0$, this gives h and it is in general impossible to adjust h' so that $\frac{\partial u}{\partial t}(x,T) = 0$. This shows that for $T < 2$, exact controllability is impossible.

We will study the problem of exact controllability for the general wave equation and we will show that for $T > 2$ we have a positive answer to the exact controllability question. Here again, we have an equation with finite speed propagation and exact controllability in time T will require a lower bound on the time T.

4. Exact Controllability for the Wave Equation

We will consider here the general problem of exact controllability for the wave equation set on a domain Ω of $\mathbb{R}^N$ and we will study extensively the case of a boundary control on a non empty part Γ_0 of the boundary Γ of Ω. Hereafter Ω will be a bounded regular open set of $\mathbb{R}^N$ and we will not discuss the case of non regular open sets.

4.1. *Exact Controllability for Boundary Control*

As the wave equation is reversible, exact controllability is equivalent to null controllability and we will study the null controllability for the following wave equation,

$$\frac{\partial^2 u}{\partial t^2} - \Delta u = 0 \quad \text{on } \Omega \times (0,T), \tag{4.1.1}$$

$$u = h \quad \text{on } \Gamma_0 \times (0,T), \tag{4.1.2}$$

$$u = 0 \quad \text{on } \Gamma \setminus \Gamma_0 \times (0,T), \tag{4.1.3}$$

$$u(0) = u_0, \quad \frac{\partial u}{\partial t}(0) = u_1. \tag{4.1.4}$$

We want to find a control h (in a suitable space) such that

$$u(T) = 0 \quad \text{and} \quad \frac{\partial u}{\partial t}(T) = 0. \tag{4.1.5}$$

First of all we have to give a sense to the solution of (4.1.1) which has non homogeneous Dirichlet data on a part of the boundary and we start with properties of the classical wave equation. We consider the wave equation

$$\frac{\partial^2 w}{\partial t^2} - \Delta w = f \quad \text{on } \Omega \times (0,T), \tag{4.1.6}$$

$$w = 0 \quad \text{on } \Gamma \times (0,T), \tag{4.1.7}$$

$$w(0) = w_0, \quad \frac{\partial w}{\partial t}(0) = w_1. \tag{4.1.8}$$

The following result is classical (see for example [8]).

Theorem 4.1.1. *If $f \in L^1(0,T;L^2(\Omega))$, $w_0 \in H_0^1(\Omega)$ and $w_1 \in L^2(\Omega)$, then there exists a unique solution w to (4.1.6) with*

$$w \in C([0,T];H_0^1(\Omega)), \quad \frac{\partial w}{\partial t} \in C([0,T];L^2(\Omega)). \tag{4.1.9}$$

Moreover, if we denote the energy by

$$E(t) = \frac{1}{2}\int_\Omega (|\frac{\partial w}{\partial t}(t)|^2 + |\nabla w(t)|^2)dx$$

we have

$$\forall t \in (0,T), \quad E(t) \le C(E(0) + |f|^2_{L^1(0,T;L^2(\Omega))}). \tag{4.1.10}$$

In particular, when $f = 0$, the energy is conserved and we have

$$\forall t \in (0,T), \quad E(t) = E(0).$$

We now have a regularity result for (4.1.6) which is often called hidden regularity result.

Theorem 4.1.2. *When Ω is regular enough and if ν denotes the unit normal vector on Γ external to Ω, when $f \in L^1(0,T;L^2(\Omega))$, $w_0 \in H_0^1(\Omega)$ and $w_1 \in L^2(\Omega)$, then we have*

$$\frac{\partial w}{\partial \nu} \in L^2(0,T;L^2(\Gamma)) \tag{4.1.11}$$

and the mapping

$$(f, w_0, w_1) \to \frac{\partial w}{\partial \nu}$$

is linear continuous from $L^1(0,T;L^2(\Omega) \times H_0^1(\Omega) \times L^2(\Omega)$ to $L^2(0,T;L^2(\Gamma))$.

Proof. In order to prove this result we use the so-called multiplier method. We denote by $(\cdot,\cdot)$ the scalar product in $L^2(\Omega)$.

Lemma 4.1.3. *Let $m \in C^1(\overline{\Omega};\mathbb{R}^N)$ be a multiplier. Then we have the following identity*

$$
\begin{aligned}
&(\frac{\partial w}{\partial t}(T), m\cdot\nabla w(T)) - (\frac{\partial w}{\partial t}(0), m\cdot\nabla w(0)) \\
&+\frac{1}{2}\int_{\Omega\times(0,T)} \operatorname{div} m(|\frac{\partial w}{\partial t}(t)|^2 - |\nabla w(t)|^2) + \sum_{i,j=1}^{N}\int_{\Omega\times(0,T)} \frac{\partial m_j}{\partial x_i}\frac{\partial w}{\partial x_i}\frac{\partial w}{\partial w}\partial x_j \\
&-\frac{1}{2}\int_{\Gamma\times(0,T)} |\frac{\partial w}{\partial \nu}|^2(m.\nu) = \int_{\Omega\times(0,T)} f(m\cdot\nabla w).
\end{aligned}
\tag{4.1.12}
$$

Proof. We have successively (the calculations are done for a set of regular dense data so that w is regular)

$$
\begin{aligned}
\int_{\Omega\times(0,T)} \frac{\partial^2 w}{\partial t^2}(m\cdot\nabla w) =& (\frac{\partial w}{\partial t}(T), m\cdot\nabla w(T)) - (\frac{\partial w}{\partial t}(0), m\cdot\nabla w(0)) \\
&-\frac{1}{2}\int_{\Omega\times(0,T)} m\cdot\nabla(|\frac{\partial w}{\partial t}|^2) \\
=& (\frac{\partial w}{\partial t}(T), m\cdot\nabla w(T)) - (\frac{\partial w}{\partial t}(0), m\cdot\nabla w(0)) \\
&+\frac{1}{2}\int_{\Omega\times(0,T)} \operatorname{div} m|\frac{\partial w}{\partial t}(t)|^2,
\end{aligned}
$$

$$
\begin{aligned}
\int_{\Omega\times(0,T)}(-\Delta w)m\cdot\nabla w =& -\int_{\Gamma\times(0,T)}\frac{\partial w}{\partial \nu}(m\cdot\nabla w) + \sum_{i,j=1}^{N}\int_{\Omega\times(0,T)}\frac{\partial m_j}{\partial x_i}\frac{\partial w}{\partial x_i}\frac{\partial w}{\partial x_j} \\
&+\sum_{i,j=1}^{N}\frac{1}{2}\int_{\Omega\times(0,T)} m_j\frac{\partial}{\partial x_j}(|\frac{\partial w}{\partial x_i}|^2) \\
=& -\int_{\Gamma\times(0,T)}\frac{\partial w}{\partial \nu}(m\cdot\nabla w) + \sum_{i,j=1}^{N}\int_{\Omega\times(0,T)}\frac{\partial m_j}{\partial x_i}\frac{\partial w}{\partial x_i}\frac{\partial w}{\partial x_j} \\
&+\frac{1}{2}\int_{\Gamma\times(0,T)}(m\cdot\nu)|\nabla w|^2 - \frac{1}{2}\int_{\Omega\times(0,T)}\operatorname{div} m|\nabla w(t)|^2 \\
=& -\frac{1}{2}\int_{\Gamma\times(0,T)}(m\cdot\nu)(\nabla w\cdot\nu)^2 \\
&+\sum_{i,j=1}^{N}\int_{\Omega\times(0,T)}\frac{\partial m_j}{\partial x_i}\frac{\partial w}{\partial x_i}\frac{\partial w}{\partial x_j} \\
&-\frac{1}{2}\int_{\Omega\times(0,T)}\operatorname{div} m|\nabla w(t)|^2.
\end{aligned}
$$

Adding up we obtain (4.1.12). □

We now go back to the proof of Theorem 4.1.2. Let us choose m as a C^1 extension in $\overline{\Omega}$ of the normal ν, so that $m \cdot \nu = 1$ on Γ. All the terms integrated in $\Omega \times (0,T)$ are bounded in terms of the energy E or the data $|f|_{L^1(0,T;L^2(\Omega))}$, $|\nabla w_0|_{L^2(\Omega)}$ and $|w_1|_{L^2(\Omega)}$. Therefore we obtain for a set of dense regular data

$$\int_{\Gamma\times(0,T)} |\frac{\partial w}{\partial \nu}|^2 \leq C(|f|^2_{L^1(0,T;L^2(\Omega))} + |\nabla w_0|^2_{L^2(\Omega)} + |w_1|^2_{L^2(\Omega)}). \quad (4.1.13)$$

Then we can extend uniquely by continuation the mapping $(f, w_0, w_1) \rightarrow \frac{\partial w}{\partial \nu}$ to a linear continuous mapping from $L^1(0,T;L^2(\Omega) \times H^1_0(\Omega) \times L^2(\Omega)$ to $L^2(0,T;L^2(\Gamma))$. □

We are now ready to define the solution of (4.1.1) by the transposition method (see for example [18]).

Theorem 4.1.4. *If $h \in L^2(0,T;L^2(\Gamma_0))$, $u_0 \in L^2(\Omega)$ and $u_1 \in H^{-1}(\Omega)$, there exists a unique solution u to (4.1.1) with*

$$u \in C([0,T];L^2(\Omega)), \quad \frac{\partial u}{\partial t} \in C([0,T];H^{-1}(\Omega)).$$

Proof. Let us first notice that when h, u_0, u_1 are taken in a dense subspace of very regular functions, the solution of (4.1.1) is classical and very regular.

For $f \in L^1(0,T;L^2(\Omega))$, $w_0 \in H^1_0(\Omega)$ and $w_1 \in L^2(\Omega)$, let w be the solution of

$$\frac{\partial^2 w}{\partial t^2} - \Delta w = f \quad \text{on } \Omega \times (0,T), \quad (4.1.14)$$

$$w = 0 \quad \text{on } \Gamma \times (0,T), \quad (4.1.15)$$

$$w(T) = w_0, \quad \frac{\partial w}{\partial t}(T) = w_1. \quad (4.1.16)$$

This equation can be reduced to (4.1.6) by changing t in $T - t$. Now, assuming u is a solution of (4.1.1), let us multiply formally (4.1.1) by w. We obtain, denoting by $\langle \cdot, \cdot \rangle$ the duality between $H^{-1}(\Omega)$ and $H^1_0(\Omega)$

$$\langle \frac{\partial u}{\partial t}(T), w_0 \rangle - \langle u_1, w(0) \rangle - (u(T), w_1) + (u_0, \frac{\partial w}{\partial t}(0))$$

$$+ \int_{\Omega\times(0,T)} uf + \int_{\Gamma\times(0,T)} u\frac{\partial w}{\partial \nu} = 0.$$

Up to now this is completely formal. Now let us define $\mathcal{L}$ by

$$\mathcal{L}(f, w_0, w_1) = -\int_{\Gamma_0\times(0,T)} h\frac{\partial w}{\partial \nu} + \langle u_1, w(0) \rangle - (u_0, \frac{\partial w}{\partial t}(0)).$$

In view of the results of Theorems 4.1.1 and 4.1.2, the mapping $\mathcal{L}$ is well defined for $f \in L^1(0,T;L^2(\Omega))$, $w_0 \in H_0^1(\Omega)$ and $w_1 \in L^2(\Omega)$ if the data satisfy $h \in L^2(0,T;L^2(\Gamma_0))$, $u_0 \in L^2(\Omega)$ and $u_1 \in H^{-1}(\Omega)$ and it is linear continuous. Therefore, from Riesz Theorem, there exist a unique triple $(u, u_0^T, u_1^T) \in L^\infty(0,T;L^2(\Omega)) \times L^2(\Omega) \times H^{-1}(\Omega)$ such that

$$\forall (f, w_0, w_1), \mathcal{L}(f, w_0, w_1) = \int_{\Omega\times(0,T)} uf + \langle u_1^T, w_0\rangle - (u_0^T, w_1). \quad (4.1.17)$$

Moreover we have

$$\begin{aligned} &|u|_{L^\infty(0,T;L^2(\Omega))} + |u_0^T|_{L^2(\Omega)} + ||u_1^T||_{H^{-1}(\Omega)} \\ &\leq C(|h|_{L^2(0,T;L^2(\Gamma_0))} + |u_0|_{L^2(\Omega)} + ||u_1||_{H^{-1}(\Omega)}). \end{aligned}$$

By taking a Cauchy sequence of regular data converging to (h, u_0, u_1) we can see that in fact $u \in C([0,T];L^2(\Omega))$. Taking various values of (f, w_0, w_1) we can interpret (4.1.18) and show that it satisfies (4.1.1) in some weak sense. It can also be shown (see [18] for a complete proof) that $\frac{\partial u}{\partial t} \in C([0,T];H^{-1}(\Omega))$ which completes the proof of Theorem 4.1.4. Anyway the correct mathematical definition of the solution is given by (4.1.17). □

There are several ways to start the study of exact controllability for (4.1.1). We will here present the Hilbert uniqueness method (HUM) introduced by J.-L. Lions in [18].

For $\varphi_0 \in H_0^1(\Omega)$ and $\varphi_1 \in L^2(\Omega)$ let φ be solution of

$$\frac{\partial^2 \varphi}{\partial t^2} - \Delta\varphi = 0 \quad \text{on } \Omega \times (0,T), \quad (4.1.18)$$

$$\varphi = 0 \quad \text{on } \Gamma \times (0,T), \quad (4.1.19)$$

$$\varphi(0) = \varphi_0, \quad \frac{\partial \varphi}{\partial t}(0) = \varphi_1. \quad (4.1.20)$$

We know that $\varphi \in C([0,T];H_0^1(\Omega)) \cap C^1([0,T];L^2(\Omega))$ and $\frac{\partial\varphi}{\partial\nu} \in L^2(0, T;L^2(\Gamma))$.

Now let ψ be the solution of

$$\frac{\partial^2 \psi}{\partial t^2} - \Delta\psi = 0 \quad \text{on } \Omega \times (0,T), \quad (4.1.21)$$

$$\psi = \frac{\partial \varphi}{\partial \nu} \quad \text{on } \Gamma_0 \times (0,T), \quad (4.1.22)$$

$$\psi = 0 \quad \text{on } \Gamma \setminus \Gamma_0 \times (0,T), \quad (4.1.23)$$

$$\psi(T) = 0, \quad \frac{\partial \psi}{\partial t}(T) = 0. \quad (4.1.24)$$

We know that $\psi \in C([0,T];L^2(\Omega)) \cap C^1([0,T];H^{-1}(\Omega))$ and it depends on φ_0 and φ_1. If we can find φ_0 and φ_1 such that

$$\psi(0) = u_0 \quad \text{and} \quad \frac{\partial \psi}{\partial t}(0) = u_1,$$

then we have solved our exact controllability problem with $h = \frac{\partial \varphi}{\partial \nu}$.

Let us call Λ the operator defined by

$$\Lambda(\varphi_0, \varphi_1) = (\frac{\partial \psi}{\partial t}(0), -\psi(0)).$$

We want to solve the equation

$$\text{Find}\, (\varphi_0, \varphi_1) \in H_0^1(\Omega) \times L^2(\Omega), \text{ such that}\, \Lambda(\varphi_0, \varphi_1) = (u_1, -u_0). \quad (4.1.25)$$

It is clear that Λ is a continuous linear map from $H_0^1(\Omega) \times L^2(\Omega)$ to $H^{-1}(\Omega) \times L^2(\Omega)$.

If we take $(\tilde{\varphi}_0, \tilde{\varphi}_1) \in H_0^1(\Omega) \times L^2(\Omega)$ and if we call $\tilde{\varphi}$ the corresponding solution of (4.1.18), we obtain after multiplication of (4.1.21) by $\tilde{\varphi}$

$$\langle \Lambda(\varphi_0, \varphi_1), (\tilde{\varphi}_0, \tilde{\varphi}_1) \rangle = \int_{\Gamma_0 \times (0,T)} \frac{\partial \varphi}{\partial \nu} \frac{\partial \tilde{\varphi}}{\partial \nu}. \quad (4.1.26)$$

From Lax-Milgram Theorem, if we have

$$\begin{aligned} E_0 = \frac{1}{2}|\nabla \varphi_0|^2_{L^2(\Omega)} + \frac{1}{2}|\varphi_1|^2_{L^2(\Omega)} &\leq C\langle \Lambda(\varphi_0, \varphi_1), (\varphi_0, \varphi_1) \rangle \\ &= C \int_{\Gamma_0 \times (0,T)} |\frac{\partial \varphi}{\partial \nu}|^2, \end{aligned}$$

then equation (4.1.25) will have a solution and our exact controllability problem will be solved.

We then have proved the following result.

Proposition 4.1.5. *If we have the observability inequality*

$$\begin{aligned} &\exists C > 0, \quad \forall (\varphi_0, \varphi_1) \in H_0^1(\Omega) \times L^2(\Omega), \\ &\frac{1}{2}|\nabla \varphi_0|^2_{L^2(\Omega)} + \frac{1}{2}|\varphi_1|^2_{L^2(\Omega)} \leq C \int_{\Gamma_0 \times (0,T)} |\frac{\partial \varphi}{\partial \nu}|^2, \end{aligned} \quad (4.1.27)$$

then we have exact controllability for equation (4.1.1) in time T with control acting on Γ_0.

In fact, it can be proved that having exact controllability with continuity of the control with respect to the data (u_0, u_1) is equivalent to the observability inequality (4.1.27).

The problem is now to find conditions on Γ_0 and T such that (4.1.27) is satisfied. Again this can be done using several methods. We will give here the result obtained by the multiplier method which has first been given in [19] and can be found in [18] or [13]. For a variant using rotated multipliers which provides some extension of the possibilities for Γ_0, see also [22].

For $x_0 \in \mathbb{R}^N$ let us define

$$\Gamma_{x_0} = \{x \in \Gamma, (x - x_0) \cdot \nu(x) > 0\} \quad \text{and} \quad R(x_0) = \max_{x \in \overline{\Omega}} |x - x_0|. \quad (4.1.28)$$

Theorem 4.1.6. *If there exists $x_0 \in \mathbb{R}^N$ such that $\Gamma_0 \supset \Gamma_{x_0}$ and if $T > 2R(x_0)$, then there exists $C > 0$ such that the observability inequality (4.1.27) is satisfied.*

Proof. We use the multiplier identity for equation (4.1.18) with $m(x) = (x - x_0)$. This gives

$$\begin{aligned} &\frac{1}{2}\int_{\Gamma \times (0,T)} (x - x_0) \cdot \nu(x) |\frac{\partial \varphi}{\partial \nu}|^2 \\ &= (\frac{\partial \varphi}{\partial t}(T), (x - x_0) \cdot \nabla\varphi(T)) - (\varphi_1, (x - x_0) \cdot \nabla\varphi_0) \\ &+ \frac{N}{2}\int_{\Omega \times (0,T)} (|\frac{\partial \varphi}{\partial t}|^2 - |\nabla\varphi|^2) + \int_{\Omega \times (0,T)} |\nabla\varphi|^2. \end{aligned}$$

This implies

$$\begin{aligned} &\frac{1}{2}\int_{\Gamma_0 \times (0,T)} (x - x_0) \cdot \nu(x) |\frac{\partial \varphi}{\partial \nu}|^2 \\ &\geq (\frac{\partial \varphi}{\partial t}(T), (x - x_0) \cdot \nabla\varphi(T)) - (\varphi_1, (x - x_0) \cdot \nabla\varphi_0) \\ &+ \frac{(N-1)}{2}\int_{\Omega \times (0,T)} (|\frac{\partial \varphi}{\partial t}|^2 - |\nabla\varphi|^2) + \frac{1}{2}\int_{\Omega \times (0,T)} (|\frac{\partial \varphi}{\partial t}|^2 + |\nabla\varphi|^2). \end{aligned}$$

On the one hand we have conservation of energy which implies

$$\frac{1}{2}\int_{\Omega \times (0,T)} (|\frac{\partial \varphi}{\partial t}|^2 + |\nabla\varphi|^2) = TE_0.$$

On the other hand, multiplying equation (4.1.18) by φ, we obtain

$$\int_{\Omega \times (0,T)} (|\frac{\partial \varphi}{\partial t}|^2 - |\nabla\varphi|^2) = (\frac{\partial \varphi}{\partial t}(T), \varphi(T)) - (\varphi_1, \varphi_0).$$

Therefore we have

$$\begin{aligned} &(\frac{\partial \varphi}{\partial t}(T), (x - x_0) \cdot \nabla\varphi(T) + \frac{(N-1)}{2}\varphi(T)) \\ &-(\varphi_1, (x - x_0) \cdot \nabla\varphi_0 + \frac{(N-1)}{2}\varphi_0) + TE_0 \\ &\leq \frac{1}{2}\int_{\Gamma_0 \times (0,T)} (x - x_0) \cdot \nu(x) |\frac{\partial \varphi}{\partial \nu}|^2 \leq \frac{1}{2}R(x_0)\int_{\Gamma_0 \times (0,T)} |\frac{\partial \varphi}{\partial \nu}|^2. \end{aligned}$$

Now for every $t \in (0,T)$ and every $\lambda > 0$ we have

$$\begin{aligned}
&|(\frac{\partial\varphi}{\partial t}(t), (x-x_0)\cdot\nabla\varphi(t) + \frac{(N-1)}{2}\varphi(t))| \\
&\leq \frac{\lambda}{2}|\frac{\partial\varphi}{\partial t}(t)|^2_{L^2(\Omega)} + \frac{1}{2\lambda}|(x-x_0)\cdot\nabla\varphi(t) + \frac{(N-1)}{2}\varphi(t)|^2_{L^2(\Omega)}.
\end{aligned}$$

But

$$\begin{aligned}
&|(x-x_0)\cdot\nabla\varphi(t) + \frac{(N-1)}{2}\varphi(t)|^2_{L^2(\Omega)} \\
&= |(x-x_0)\cdot\nabla\varphi(t)|^2_{L^2(\Omega)} + (N-1)\int_\Omega (x-x_0)\cdot\nabla\varphi(t)\varphi(t) \\
&\quad + \frac{(N-1)^2}{4}|\varphi(t)|^2_{L^2(\Omega)} \\
&= |(x-x_0)\cdot\nabla\varphi(t)|^2_{L^2(\Omega)} - \frac{(N-1)}{2}\frac{(N+1)}{2}|\varphi(t)|^2_{L^2(\Omega)} \\
&\leq |(x-x_0)\cdot\nabla\varphi(t)|^2_{L^2(\Omega)} \leq R^2(x_0)|\nabla\varphi(t)|^2_{L^2(\Omega)}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&|(\frac{\partial\varphi}{\partial t}(t), (x-x_0)\cdot\nabla\varphi(t) + \frac{(N-1)}{2}\varphi(t))| \\
&\leq \frac{\lambda}{2}|\frac{\partial\varphi}{\partial t}(t)|^2_{L^2(\Omega)} + \frac{R^2(x_0)}{2\lambda}|\nabla\varphi(t)|^2_{L^2(\Omega)}.
\end{aligned}$$

Taking $\lambda = R(x_0)$ we obtain for every $t \in (0,T)$

$$|(\frac{\partial\varphi}{\partial t}(t), (x-x_0)\cdot\nabla\varphi(t) + \frac{(N-1)}{2}\varphi(t))| \leq R(x_0)E_0.$$

We then have

$$(T - 2R(x_0))E_0 \leq \frac{1}{2}R(x_0)\int_{\Gamma_0\times(0,T)}|\frac{\partial\varphi}{\partial\nu}|^2$$

so that if $T > 2R(x_0)$ we obtain

$$E_0 \leq \frac{R(x_0)}{2(T-2R(x_0))}\int_{\Gamma_0\times(0,T)}|\frac{\partial\varphi}{\partial\nu}|^2$$

and this gives us the observability inequality with explicit constants. □

Remark 4.1.7. 1) *The minimum time $T(x_0) = 2R(x_0)$ that we obtain above is not optimal.*

2) *The above theorem says that Γ_0 has to be large enough.For example, if Ω is a disk for $N = 2$, we need Γ_{x_0} to be larger than half the circle.*

3) *We can also use Carleman estimates to prove the observability inequality and this gives the same conditions as the multiplier method.*

4) *It has been proved by Bardos-Lebeau-Rauch (see [1]), using microlocal analysis arguments, that the observability inequality is true when* Γ_0 *and* T *satisfy the so-called Geometric Control Condition (GCC) which says that for every* $x \in \Omega$, *every ray of the geometrical optics travelling at speed* 1 *(and reflecting on the boundary) meets* Γ_0 *before time* T *at a non diffractive point.*

This condition has been proved to be necessary and sufficient by Burq-Gerard in [4] using defect measures.

It can be seen that if Ω *is a disk, then (GCC) implies that for each diameter, at least one extremity is in* Γ_0.

4.2. *Case of Distributed Control*

We can study the case of distributed control for the wave equation. Let ω be a non empty open subset of Ω and let χ_ω be the characteristic function of ω. We consider the following wave equation where the control acts on ω

$$\frac{\partial^2 u}{\partial t^2} - \Delta u = h\chi_\omega \quad \text{on } \Omega \times (0,T), \tag{4.2.29}$$

$$u = 0 \quad \text{on } \Gamma \times (0,T), \tag{4.2.30}$$

$$u(0) = u_0, \quad \frac{\partial u}{\partial t}(0) = u_1. \tag{4.2.31}$$

We already know that for $h \in L^2(0,T;L^2(\omega))$, $u_0 \in H_0^1(\Omega)$ and $u_1 \in L^2(\Omega)$ there exists a unique solution u satisfying

$$u \in C([0,T];H_0^1(\Omega)) \cap C^1([0,T];L^2(\Omega)).$$

The question of exact controllability (or null controllability) is here to find $h \in L^2(0,T;L^2(\omega))$ such that we have at time T

$$u(T) = 0 \quad \text{and} \quad \frac{\partial u}{\partial t}(T) = 0.$$

For $\epsilon > 0$ let us define

$$\omega_{\epsilon,x_0} = \cup_{x\in\Gamma_{x_0}}(B(x,\epsilon) \cap \Omega) \tag{4.2.32}$$

where Γ_{x_0} is defined in (4.1.28). The following result is proved in [18].

Theorem 4.2.1. *Let* ω *be such that there exists* $x_0 \in \mathbb{R}^N$ *and* $\epsilon > 0$ *such that* $\omega_{\epsilon,x_0} \subset \omega$ *and let* T *be given such that* $T > 2R(x_0)$. *Then for every* $u_0 \in H_0^1(\Omega)$ *and* $u_1 \in L^2(\Omega)$, *there exists* $h \in L^2(0,T;L^2(\omega))$ *such that the solution* u *of (4.2.29) satisfies*

$$u(T) = 0 \quad \text{and} \quad \frac{\partial u}{\partial t}(T) = 0.$$

In fact here exact controllability is a consequence of the following observability inequality on the adjoint equation (4.1.18). We have to prove that there exists a constant $C > 0$ such that

$$\forall \varphi_0 \in L^2(\Omega), \quad \varphi_1 \in H^{-1}(\Omega), \quad |\varphi_0|^2_{L^2(\Omega)} + ||\varphi_1||^2_{H^{-1}(\Omega)} \leq C \int_{\omega \times (0,T)} |\varphi|^2.$$

This proof is technical and uses the observability inequality for the case of boundary control.

In [1] it is proved that exact controllability holds if ω satisfies the Geometric Control Condition saying that every ray of the geometrical optics meets ω before time T.

5. Controllability of Schrödinger Equation

We give here a rapid presentation of the controllability for the free Schrödinger equation in a bounded domain with boundary control. This is a first step concerning Schrödinger equation as the most relevant problem consists in controlling the equation via the action of a potential but this is a bilinear control problem which is quite different and has been the object of recent studies, for example in [2], [3] or [24].

5.1. *Schrödinger Equation*

We keep the notations of the previous section and we consider the free Schrödinger equation with control on a part of the boundary

$$i\frac{\partial u}{\partial t} + \Delta u = 0 \quad \text{in } \Omega \times (0,T), \tag{5.1.1}$$

$$u = h \quad \text{on } \Gamma_0 \times (0,T), \tag{5.1.2}$$

$$u = 0 \quad \text{on } \Gamma \setminus \Gamma_0 \times (0,T), \tag{5.1.3}$$

$$u(0) = u_0 \quad \text{in } \Omega. \tag{5.1.4}$$

Again here using the transposition method we can prove the following existence result.

Proposition 5.1.1. *Let Ω be a bounded open set of $\mathbb{R}^N$ of class C^3 with $\alpha > 0$. For any $u_0 \in H^{-1}(\Omega)$ and $h \in L^2(0,T;L^2(\Gamma_0))$, there exists a unique solution u to (5.1.1) with*

$$u \in C([0,T];H^{-1}(\Omega)).$$

5.2. *Controllability Results*

For the study of controllability for (5.1.1) we first remark that here again the equation is reversible and exact controllability is equivalent to null controllability. Then using the Hilbert uniqueness method as for the wave equation, null controllability (with continuity of the control) is equivalent to an observability inequality for the following adjoint problem. We consider the solution φ to the free Schrödinger equation

$$i\frac{\partial\varphi}{\partial t} + \Delta\varphi = 0 \quad \text{in } \Omega \times (0,T), \tag{5.2.5}$$

$$\varphi = 0 \quad \text{on } \Gamma \times (0,T), \tag{5.2.6}$$

$$\varphi(0) = \varphi_0 \quad \text{in } \Omega. \tag{5.2.7}$$

We want to prove the following observability inequality

$$\exists C > 0, \quad \forall \varphi_0 \in H_0^1(\Omega), \quad ||\varphi_0||^2_{H_0^1(\Omega)} \leq C \int_{\Gamma_0 \times (0,T)} |\frac{\partial\varphi}{\partial\nu}|^2. \tag{5.2.8}$$

Using the multiplier method in a similar way as it is done for the wave equation, E. Machtyngier proved in [20] that (4.1.28) is true when Γ_0 contains a set Γ_{x_0} as defined in (4.1.28) and for any $T > 0$. Therefore she obtains the following result.

Theorem 5.2.1. *Let Γ_0 be such that there exists $x_0 \in \mathbb{R}^N$ such that Γ_{x_0} and let $T > 0$ be given. Then for every $u_0 \in H^{-1}(\Omega)$, there exists $h \in L^2(0,T;L^2(\Gamma_0))$ such that the solution u of (5.1.1) satisfies*

$$u(T) = 0.$$

Using microlocal analysis arguments, G. Lebeau in [16] extended this result to the case of Γ_0 satisfying the Geometric Control Condition saying that every ray of the geometrical optics reaches Γ_0 at a non diffractive point in uniform time.

More recently, in [26], the authors proved that in dimension $N = 2$ for the case of a rectangle Ω, the exact controllability result holds as soon as Γ_0 contains at least one interval in each direction of the axis.

6. Controllability of Linear Diffusion Convection Equations

In this section we will study the controllability of linear diffusion convection equation, the simplest model being the heat equation. These equations are not reversible and therefore null controllability is no longer equivalent to exact controllability. On the other hand, for the heat equation for example, the solution can become very regular for positive time and the regularity of the reachable set is very difficult to describe. Therefore the study of exact controllability is not relevant here and we will restrict ourselves to the study of exact controllability to trajectories which has been described earlier. As we deal here with linear operators, this notion is equivalent to null controllability, but we always have to keep in mind that this is only a first step towards the study of controllability for nonlinear equations and in this context, the notion of exact controllability to trajectories seems to be the good one.

For simplicity of the presentation, we will restrict ourselves to the case of distributed control.

6.1. *Statement of the Problem and Result*

Let $T > 0$ and let Ω be a bounded open regular subset of $\mathbb{R}^N$. We denote by Γ the boundary of Ω which is supposed to be of class $C^{2+\alpha}$. We consider an operator L, which may depend on time t, which is elliptic for each time t and which is defined by

$$Lz = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}(a_{i,j}\frac{\partial z}{\partial x_j}) + \sum_{i=1}^{N} \frac{\partial}{\partial x_i}(b_i z) + a_0 z, \tag{6.1.1}$$

where

$$\forall i,j = 1,\cdots,N, \quad a_{i,j} \in W^{1,\infty}(\Omega \times (0,T)), \quad a_{i,j} = a_{j,i}, \tag{6.1.2}$$

$$\forall i = 1,\cdots,N, \quad b_i, a_0 \in L^{\infty}(\Omega \times (0,T)) \tag{6.1.3}$$

and the coefficients $a_{i,j}$ satisfy an ellipticity condition uniformly in t

$$\exists \beta > 0, \quad \forall (x,t) \in \Omega \times (0,T), \quad \forall \xi \in \mathbb{R}^N, \quad \sum_{i,j=1}^{N} a_{i,j}(x,t)\xi_j \xi_i \geq \beta|\xi|^2. \tag{6.1.4}$$

Let ω be a non empty open subset of Ω and χ_ω be its characteristic function. For each control $v \in L^2(0,T;L^2(\omega))$ we consider the following controlled diffusion-convection equation

$$\frac{\partial y}{\partial t} + Ly = f^0 + v \cdot \chi_\omega \quad \text{in } \Omega \times (0,T), \tag{6.1.5}$$

$$y = 0 \quad \text{on } \Gamma \times (0,T), \tag{6.1.6}$$

$$y(x,0) = y^0(x) \quad \text{in } \Omega, \tag{6.1.7}$$

where $y^0 \in L^2(\Omega)$ and $f^0 \in L^2(0,T;H^{-1}(\Omega))$ for example.

It is well known (cf. for example [8]) that for every $v \in L^2(0,T;L^2(\omega))$ there exists a unique solution $y = y(v)$ to equation (6.1.5) with $y \in C([0,T];L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$.

Let us now consider an "ideal" uncontrolled trajectory starting at time $t = 0$ from the initial data $\overline{y}^0 \in L^2(\Omega)$

$$\frac{\partial \overline{y}}{\partial t} + L\overline{y} = f^0 \quad \text{in } \Omega \times (0,T), \tag{6.1.8}$$

$$\overline{y} = 0 \quad \text{on } \Gamma \times (0,T), \tag{6.1.9}$$

$$\overline{y}(x,0) = \overline{y}^0(x) \quad \text{in } \Omega. \tag{6.1.10}$$

Again we have a unique solution for (6.1.8) $\overline{y} \in C([0,T];L^2(\Omega)) \cap L^2(0,T;L^2(\Omega))$. The question of exact controllability to trajectories is then, for every $y^0 \in L^2(\Omega)$ and $\overline{y}^0 \in L^2(\Omega)$, to find $v \in L^2(0,T;L^2(\omega))$ such that $y(T) = \overline{y}(T)$.

Remark 6.1.1. 1) *We have taken here the same right-hand side f^0 in both equations (6.1.5) and (6.1.8) for sake of simplicity. We could have taken in (6.1.8) a right-hand side $\overline{f}^0 \neq f^0$ but the conditions we would have to impose on $f^0 - \overline{f}^0$ are not easy to state correctly. Nevertheless the proof will be given with $g^0 = f^0 - \overline{f}^0 \neq 0$.*

2) *We have taken here the Dirichlet boundary conditions. Other types of boundary conditions can be considered, for example Neumann conditions or Fourier conditions, see [11] and [10].*

Of course, as already mentioned above, as we deal here with a linear equation, the problem is equivalent to the following null-controllability one. Let z be the solution to

$$\frac{\partial z}{\partial t} + Lz = v \cdot \chi_\omega \quad \text{in } \Omega \times (0,T), \tag{6.1.11}$$

$$z = 0 \quad \text{on } \Gamma \times (0,T), \tag{6.1.12}$$

$$z(x,0) = z^0(x) \quad \text{in } \Omega, \tag{6.1.13}$$

where $z^0 \in L^2(\Omega)$. We then look for $v \in L^2(0,T;L^2(\omega))$ such that

$$z(T) = 0. \tag{6.1.14}$$

The main result of this section is the following

Theorem 6.1.2. *Under the previous hypotheses (6.1.2), (6.1.3), (6.1.4), for every open subset ω of Ω, for every time $T > 0$ and for every $z^0 \in L^2(\Omega)$, there exists a control $v \in L^2(0,T;L^2(\omega))$ such that (6.1.14) holds. Moreover we can obtain a control v of minimal norm in $L^2(0,T;L^2(\omega))$ among the admissible controls (such that (6.1.14) is satisfied).*

The proof of Theorem 6.1.2 will require several steps which will be developed in the next sections. There are various strategies which lead to the result, all based on Carleman inequalities. The first results were obtained independently by Lebeau-Robbiano in [15] for the pure heat equation and by Fursikov-Imanuvilov in [11] for the general case. We will present here the method of [11] with two different strategies. First of all, we will develop a method starting from an optimal control problem because it is quite natural, adaptable to many other situations and also easier to understand. We will then give a second approach which turns out to be useful for extension to nonlinear problems.

Remark 6.1.3. *For some further extensions such as treating the case of nonlinear diffusion-convection equations, we sometimes need to obtain a control v in a smaller space like $L^r(O,T;L^q(\Omega))$ with r and q larger than* 2*. We can obtain the previous theorem with this class of controls but this requires some minor modifications in the proof for example a careful use of regularity results for the heat equation.*

6.2. *An Auxiliary Optimal Control Problem*

We now take $g^0 \in L^2(0,T;L^2(\Omega))$ for the moment (further conditions will be needed later on) and we consider the following variant of (6.1.11)

$$\frac{\partial z}{\partial t} + Lz = g^0 + v \cdot \chi_\omega \quad \text{in } \Omega \times (0,T), \tag{6.2.15}$$

$$z = 0 \quad \text{on } \Gamma \times (0,T), \tag{6.2.16}$$

$$z(x,0) = z^0(x) \quad \text{in } \Omega. \tag{6.2.17}$$

Let ϵ be a strictly positive number. We define the functional

$$J_\epsilon(v) = \frac{1}{2\epsilon}|z(T)|^2_{L^2(\Omega)} + \frac{1}{2}\int_{\omega\times(0,T)} |v|^2. \tag{6.2.18}$$

We want to study in a first step the following optimal control problem

$$\min_{v \in L^2(0,T;L^2(\omega))} J_\epsilon(v). \tag{6.2.19}$$

This is a natural approximation to our null controllability problem as, for ϵ very small, the first term in J_ϵ (at the minimum) should force the value $z(T)$ to be small.

Proposition 6.2.1. *The optimal control problem has a unique solution $v_\epsilon \in L^2(0,T;L^2(\omega))$.*

If L^* is the adjoint operator of L defined by

$$L^*\xi = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_j}(a_{i,j}\frac{\partial \xi}{\partial x_i}) - \sum_{i=1}^{N} b_i \frac{\partial \xi}{\partial x_i} + a_0\xi, \tag{6.2.20}$$

this solution v_ϵ is characterized by the following optimality system.

$$\frac{\partial z_\epsilon}{\partial t} + Lz_\epsilon = g^0 + v_\epsilon \cdot \chi_\omega \quad \text{in } \Omega \times (0,T), \tag{6.2.21}$$

$$z_\epsilon = 0 \quad \text{on } \Gamma \times (0,T), \tag{6.2.22}$$

$$z_\epsilon(x,0) = z^0(x) \quad \text{in } \Omega, \tag{6.2.23}$$

$$-\frac{\partial \xi_\epsilon}{\partial t} + L^*\xi_\epsilon = 0 \quad \text{in } \Omega \times (0,T), \tag{6.2.24}$$

$$\xi_\epsilon = 0 \quad \text{on } \Gamma \times (0,T), \tag{6.2.25}$$

$$\xi_\epsilon(T) = \frac{1}{\epsilon} z_\epsilon(T) \quad \text{in } \Omega, \tag{6.2.26}$$

$$\xi_\epsilon + v_\epsilon = 0 \quad \text{in } \omega \times (0,T). \tag{6.2.27}$$

Proof. The functional J_ϵ is well defined, continuous and strictly convex and therefore, it is classical that the optimal control problem has a unique solution v_ϵ (see [17]). The necessary and sufficient optimality condition says that

$$DJ_\epsilon(v_\epsilon)[w] = 0, \quad \forall w \in L^2(0,T;L^2(\omega)).$$

In order to compute the derivative of J_ϵ we have to define the derivative of z with respect to v and we have

$$Dz(v)[w] = \psi$$

where ψ satisfies

$$\frac{\partial \psi}{\partial t} + L\psi = w \cdot \chi_\omega \quad \text{in } \Omega \times (0,T), \tag{6.2.28}$$

$$\psi = 0 \quad \text{on } \Gamma \times (0,T), \tag{6.2.29}$$

$$\psi(x,0) = 0 \quad \text{in } \Omega. \tag{6.2.30}$$

We now have

$$DJ_\epsilon(v_\epsilon)[w] = (\frac{1}{\epsilon} z_\epsilon(T), \psi(T)) + \int_{\omega \times (0,T)} v_\epsilon \cdot w.$$

Multiplying equation for ψ by ξ_ϵ solution of (6.2.24) we obtain

$$(\frac{1}{\epsilon} z_\epsilon(T), \psi(T)) = \int_{\omega \times (0,T)} \xi_\epsilon \cdot w.$$

Therefore the optimality condition says that

$$\int_{\omega\times(0,T)} (v_\epsilon + \xi_\epsilon)\cdot w, \quad \forall w \in L^2(0,T;L^2(\omega)),$$

and this is equivalent to

$$v_\epsilon + \xi_\epsilon = 0 \quad \text{in } \omega\times(0,T).$$

This completes the proof of Proposition 6.2.1. □

6.3. *Null Controllability Modulo Observability Inequality*

Let us now multiply equation (6.2.21) for z_ϵ by ξ_ϵ solution of (6.2.24). We obtain

$$\frac{1}{\epsilon}|z_\epsilon(T)|^2_{L^2(\Omega)} - \int_{\omega\times(0,T)} \xi_\epsilon\cdot v_\epsilon = (z_0,\xi_\epsilon(0)) + \int_{\Omega\times(0,T)} g^0\cdot\xi_\epsilon.$$

Notice that we have

$$-\int_{\omega\times(0,T)} \xi_\epsilon\cdot v_\epsilon = \int_{\omega\times(0,T)} |v_\epsilon|^2 = \int_{\omega\times(0,T)} |\xi_\epsilon|^2,$$

which implies that

$$2J_\epsilon(v_\epsilon) = \frac{1}{\epsilon}|z_\epsilon(T)|^2_{L^2(\Omega)} + \int_{\omega\times(0,T)} |\xi_\epsilon|^2 = (z_0,\xi_\epsilon(0)) + \int_{\Omega\times(0,T)} g^0\cdot\xi_\epsilon.$$

Let us introduce a weight ρ, which will be precisely defined later on, but which, for the moment, only satisfies $\rho > 0$ in $\Omega\times(0,T)$. Using Hölder inequality we obtain provided that

$$\rho g^0 \in L^2(0,T;L^2(\Omega)),$$

$$\frac{1}{\epsilon}|z_\epsilon(T)|^2_{L^2(\Omega)} + \int_{\omega\times(0,T)} |\xi_\epsilon|^2 \qquad (6.3.31)$$

$$\le 2(|z^0|^2_{L^2(\Omega)} + |\rho g^0|^2_{L^2(0,T;L^2(\Omega))})^{\frac{1}{2}}(|\xi_\epsilon(0)|^2_{L^2(\Omega)} + |\frac{1}{\rho}\xi_\epsilon|^2_{L^2(0,T;L^2(\Omega))})^{\frac{1}{2}}.$$

We would like to deduce from this last inequality a bound for

$$|v_\epsilon|^2_{L^2(0,T;L^2(\omega))} = \int_{\omega\times(0,T)} |\xi_\epsilon|^2.$$

This will be the case if for some suitable weight ρ there exists a constant C (independent of ϵ) such that we have the following inequality

$$|\xi_\epsilon(0)|^2_{L^2(\Omega)} + |\frac{1}{\rho}\xi_\epsilon|^2_{L^2(0,T;L^2(\Omega))} \le C\int_{\omega\times(0,T)} |\xi_\epsilon|^2. \qquad (6.3.32)$$

Such an inequality will be called an observability inequality and its proof will be the subject of the next sections. Let us for the moment suppose that inequality (6.3.32) holds true. Then we easily obtain that for every $\epsilon > 0$

$$|v_\epsilon|^2_{L^2(0,T;L^2(\omega))} = \int_0^T \int_\omega |\xi_\epsilon|^2 \le C(|z^0|^2_{L^2(\Omega)} + |\rho g^0|^2_{L^2(0,T;L^2(\Omega))}) \quad (6.3.33)$$

and

$$\frac{1}{\epsilon}|z_\epsilon(T)|^2_{L^2(\Omega)} \le C(|z^0|^2_{L^2(\Omega)} + |\rho g^0|^2_{L^2(0,T;L^2(\Omega))}). \quad (6.3.34)$$

Therefore v_ϵ is bounded in $L^2(0,T;L^2(\omega))$ independently of ϵ and for a subsequence (still denoted by ϵ) we have

$$v_\epsilon \rightharpoonup v \quad \text{in } L^2(0,T;L^2(\omega)) \text{ weakly.}$$

Consequently, we have for example

$$z_\epsilon \rightharpoonup z(v) \quad \text{in } C([0,T];L^2(\Omega)) \text{ weakly,}$$

where $z(v)$ is the solution of (6.2.15) associated to v. As $z_\epsilon(T)$ converges to 0 in $L^2(\omega)$ we must have

$$z(v)(T) = 0$$

and this completes the first part of Theorem 6.1.2.

Now we know that $J_\epsilon(v_\epsilon) \le J_\epsilon(v) = \frac{1}{2}\int_{\omega\times(0,T)} |v|^2$, therefore

$$\frac{1}{2}\int_{\omega\times(0,T)} |v_\epsilon|^2 \le \frac{1}{2}\int_{\omega\times(0,T)} |v|^2.$$

This implies that

$$v_\epsilon \to v \quad \text{in } L^2(0,T;L^2(\omega)) \text{ strongly}$$

and that if $\tilde{v}$ is another admissible control (such that null controllability is achieved for $\tilde{v}$), then

$$\frac{1}{2}\int_{\omega\times(0,T)} |v|^2 \le \frac{1}{2}\int_{\omega\times(0,T)} |\tilde{v}|^2.$$

This completes the proof of Theorem 6.1.2 if we know the observability inequality (6.3.32). □

Remark 6.3.1. *In fact, assuming that the observability inequality (6.3.32) is valid we have proved a stronger result as we have taken in (6.2.15) $g^0 \neq 0$ but we have to assume that $\rho g^0 \in L^2(0,T;L^2(\Omega))$.*

Now, in order to finish the proof of Theorem 6.1.2, it remains to show the observability inequality (6.3.32) and to choose of course a suitable weight ρ. Such an inequality says that the knowledge of the solution of (6.2.15) on a small cylinder $\omega \times (0,T)$ governs the (weighted) behaviour of the solution on the whole domain $\Omega \times (0,T)$ and its final value (notice that (6.2.15) is a backward equation so that the final value is $z(0)$). This must require a way to propagate the information inside the spacial domain and will be a consequence of a global Carleman inequality, the proof of which will be the object of the next section.

6.4. *Global Carleman Inequality*

Local Carleman estimates have been introduced by Carleman in [5] to study uniqueness problems, but we will here require global Carleman estimates following [11]. We first have to define the weight function that we will use. There are several possible choices and we follow here the 2-parameters choice of [11] with some slight modification.

6.4.1. *Weight Functions*

We have to choose a weight which we called ρ in the last section but this choice will require long arguments and we have to begin with a basic choice of weight depending only on the space variables. This weight is fundamental in the sense that, roughly speaking, information will propagate in space along the gradient lines of this function.

Lemma 6.4.1. *Let ω_0 be an open set such that $\overline{\omega_0} \subset \omega$ (for example ω_0 can be a small open ball). Then there exists $\psi \in C^2(\overline{\Omega})$ such that*

$$\begin{aligned} \psi(x) > 0, &\quad \forall x \in \Omega, \\ \psi(x) = 0, &\quad \forall x \in \Gamma, \\ |\nabla\psi(x)| \neq 0, &\quad \forall x \in \overline{\Omega} - \omega_0. \end{aligned}$$

Proof. This proof is very technical and can be omitted in a first step.

As Ω is regular, we can first choose a function $\theta \in C^2(\mathbb{R}^N)$ such that $\Omega = \{x \in \mathbb{R}^N,\ \theta(x) > 0\}$ and $|\nabla\theta(x)| \neq 0,\ \forall x \in \Gamma$. This can be done locally, and then extended globally using a partition of unity. From Morse's density theorem, there exists a sequence of Morse functions (θ_k) (i.e. such that their gradients vanishes only at a finite number of points) such that $\theta_k \to \theta$ in $C^2(\overline{\Omega})$ when $k \to +\infty$ (θ_k does not necessary vanish on the boundary). Moreover we can take $\theta_k > 0$ as $\theta > 0$ on Ω. Let us define

$C = \{x \in \mathbb{R}^N, \nabla\theta(x) = 0\}$ as the set of critical points of θ. As $|\nabla\theta(x)| \neq 0,\ \forall x \in \Gamma$, there exists an open neighborhood V of Γ in $\mathbb{R}^N$ and $\delta > 0$ such that

$$\forall x \in \overline{V}, \quad |\nabla\theta(x)| \geq \delta.$$

Let $\varphi \in C_0^\infty(V)$ such that $\varphi(x) = 1, \forall x \in \Gamma$ and $0 \leq \varphi \leq 1$. We set

$$\mu_k(x) = \theta_k(x) + \varphi(x)(\theta(x) - \theta_k(x)).$$

Then $\mu_k(x) = 0,\ \forall x \in \Gamma$, $\mu_k(x) > 0,\ \forall x \in \Omega$ and moreover

$$\forall x \in \overline{\Omega - V}, \quad \nabla\mu_k(x) = \nabla\theta_k(x).$$

Now if $x \in \Omega \cap V$, we have

$$\nabla\mu_k(x) = \nabla\theta_k(x) + \varphi(x)(\nabla\theta(x) - \nabla\theta_k(x)) + \nabla\varphi(x)(\theta(x) - \theta_k(x))$$

so that for $k \geq k_0$, k_0 large enough we have

$$\begin{aligned} |\nabla\mu_k(x)| &\geq |\nabla\theta_k(x)| - 2||\varphi||_{C^1(\overline{\Omega})}||\theta - \theta_k||_{C^1(\overline{\Omega})} \\ &\geq \delta - 2||\varphi||_{C^1(\overline{\Omega})}||\theta - \theta_k||_{C^1(\overline{\Omega})} \\ &\geq \frac{\delta}{2}. \end{aligned}$$

Let us choose $k \geq k_0$ and set $\mu(x) = \mu_k(x)$. Then μ is a Morse function because the points where its gradient vanishes are among the points where $\nabla\theta_k$ vanishes. Moreover we have $\mu(x) = 0,\ \forall x \in \Gamma$.

Let now $x_1, x_2, \cdots, x_r$ be the critical points of μ. Then for $i = 1, \cdots, r$ we have $x_i \in \Omega - \overline{V}$. We can find r disjoint regular paths $l_1, \cdots, l_r$ such that for $i = 1, \cdots, r$,

$$\begin{aligned} &l_i \in C^\infty([0,1]; \mathbb{R}^N), \\ &l_i(t) \in \Omega - \overline{V}, \quad \forall t \in [0,1], \\ &l_i(t_1) \neq l_i(t_2), \quad \forall t_1, t_2 \in [0,1], \quad t_1 \neq t_2, \\ &l_i(1) = x_i \quad \text{and} \quad l_i(0) \in \omega_0, \\ &\forall s, t \in [0,1], \quad l_i(s) \neq l_j(t), \quad \text{when } i \neq j, \end{aligned}$$

and we can find r functions $f_1, \cdots, f_r$ such that for $i = 1, \cdots, r$

$$f_i \in C^\infty(\mathbb{R}^N, \mathbb{R}^N) \quad \text{and} \quad \frac{dl_i}{dt}(t) = f_i(l_i(t)), \quad \forall t \in (0,1).$$

Now, for $i = 1, \cdots, r$ we can find open neighborhoods W_i of the sets $\{l_i(t),\ t \in [0,1]\}$ such that

$$W_i \subset \Omega - \overline{V} \quad \text{and} \quad W_i \cap W_j = \emptyset \text{ if } i \neq j.$$

Then we take functions $e_i \in \mathcal{D}(W_i)$ such that $e_i(l_i(t)) = 1,\ \forall t \in [0,1]$ and we set

$$g_i(x) = e_i(x) f_i(x).$$

Let us consider the differential equation

$$\frac{dx}{dt}(t) = g_i(x(t)), \quad \forall t \in (0,1),$$
$$x(0) = x.$$

We denote by $\mathcal{S}_t^i : \mathbb{R}^N \to \mathbb{R}^N$ the operator which maps x to $x(t)$. We then have

$$\mathcal{S}_1^i(l_i(0)) = x_i, \quad i = 1, \cdots, r.$$

We now define

$$\mathcal{S}(x) = \mathcal{S}_1^1 \circ \mathcal{S}_1^2 \circ \cdots \circ \mathcal{S}_1^r(x).$$

We can se that if $x \in \Omega - (\bigcup_{i=1}^r W_i)$, then $\mathcal{S}(x) = x$ and therefore

$$\forall x \in V, \quad \mathcal{S}(x) = x.$$

On the other hand, each $\mathcal{S}_1^i$ is a diffeomorphism from Ω into itself, so is $\mathcal{S}$ and $\nabla \mathcal{S}$ is invertible.

Let us now set

$$\psi(x) = \mu(\mathcal{S}(x)).$$

Then we have $\psi(x) = 0,\ \forall x \in \Gamma$. Moreover, as $\nabla \mathcal{S}$ is invertible, if $\nabla \psi(x) = 0$, this means that $\mathcal{S}(x) \in \{x_1, \cdots, x_r\}$. But we know that $\mathcal{S}_1^j = Id$ on $\Omega - W_j$ so that

$$\mathcal{S}(l_i(0)) = \mathcal{S}_1^i(l_i(0)) = x_i.$$

As $\mathcal{S}$ is a diffeomorphism, we see that

$$\mathcal{S}(x) \in \{x_1, \cdots, x_r\} \Rightarrow x \in \{l_1(0), \cdots, l_r(O)\} \Rightarrow x \in \omega_0.$$

Therefore

$$\nabla \psi(x) = 0 \Rightarrow x \in \omega_0,$$

and ψ satisfies all conditions of the lemma. This finishes the proof of Lemma 6.4.1.

We will now use the function ψ given by Lemma 6.4.1 to build new weights. Let us define for $\lambda > 0$ and for an integer $k \geq 1$

$$\varphi(x,t) = \frac{e^{\lambda(\psi(x)+m_1)}}{t^k(T-t)^k}, \tag{6.4.35}$$

$$\eta(x,t) = \frac{e^{\lambda(|\psi|_{L^\infty(\Omega)}+m_2)} - e^{\lambda(\psi(x)+m_1)}}{t^k(T-t)^k}, \tag{6.4.36}$$

where the positive constants m_1 and m_2 will be chosen below.

Remark 6.4.2. *For our purpose here, we only need to take $k = 1$ but for further extensions it happens that we sometimes need to take $k > 1$ and this does not make any change in the sequel.*

We now want to choose the constants m_1 and m_2 so that the numerator of η is positive which implies that $m_2 > m_1$ and also that we can bound (modulo some constants) $\frac{\partial \eta}{\partial t}$ and $\frac{\partial^2 \eta}{\partial t^2}$ respectively by φ^2 and φ^3. A simple calculation shows that for example, a possible choice of these constants is

$$m_1 = |\psi|_{L^\infty(\Omega)} + 2; \quad m_2 = |\psi|_{L^\infty(\Omega)} + 3. \tag{6.4.37}$$

We now have for every $\lambda > 0$ the following properties which will be helpful for our calculations

$$\nabla\varphi = \lambda\varphi\nabla\psi, \quad \nabla\eta = -\lambda\varphi\nabla\psi, \tag{6.4.38}$$

$$1 \leq (\frac{T}{2})^{2k}\varphi, \quad \varphi \leq (\frac{T}{2})^{2k}\varphi^2, \quad \varphi \leq (\frac{T}{2})^{4k}\varphi^3, \tag{6.4.39}$$

$$|\frac{\partial\varphi}{\partial t}| \leq kT(\frac{T}{2})^{2(k-1)}\varphi^2, \quad |\frac{\partial^2\varphi}{\partial t^2}| \leq k(k+1)T^2(\frac{T}{2})^{4(k-1)}\varphi^3, \tag{6.4.40}$$

$$|\frac{\partial\eta}{\partial t}| \leq kT(\frac{T}{2})^{2(k-1)}\varphi^2, \quad |\frac{\partial^2\eta}{\partial t^2}| \leq k(k+1)T^2(\frac{T}{2})^{4(k-1)}\varphi^3. \tag{6.4.41}$$

We can notice that η tends to $+\infty$ when $t \to T$ or $t \to 0$ but that η is uniformly bounded in $\Omega \times [\delta, T-\delta]$ if $\delta > 0$.

Our final weight will depend on a second positive parameter s and will be of the form $e^{-s\eta(x,t)}$. We can see that, for fixed s, this function tends very rapidly to 0 when $t \to T$ or $t \to 0$.

6.4.2. *Proof of a Global Carleman Inequality*

We want to prove a Carleman inequality for the solution of equation (6.2.24) but we will take the general case of a parabolic equation. We still consider a backward equation because (6.2.24) is backward but of course there is no change for a forward equation. As we will see we only need to consider the principal part of L^* and as the coefficients $a_{i,j}$ are symmetric it is equivalent to take the principal part of L that we call L_0. So we define

$$L_0 u = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}(a_{i,j}\frac{\partial u}{\partial x_j}), \tag{6.4.42}$$

and for $g \in L^2(0,T;L^2(\Omega))$ we consider the solution u of the following backward parabolic equation

$$-\frac{\partial u}{\partial t} + L_0 u = g \quad \text{in } \Omega \times (0,T), \tag{6.4.43}$$

$$u = 0 \quad \text{on } \Gamma \times (0,T), \tag{6.4.44}$$

$$u(T) = u_0 \quad \text{in } \Omega. \tag{6.4.45}$$

We can now state the global Carleman inequality

Theorem 6.4.3. *There exist parameters $s_0 > 0$ and $\lambda_0 > 0$ and there exists a constant $C > 0$ depending only on Ω, ω_0, ψ, on β defined in (6.1.4) and on the coefficients $a_{i,j}$ such that for every $s > T^{2k}s_0$, for every $\lambda > \lambda_0$ and for every solution of (6.4.43) we have*

$$\begin{aligned} &\frac{1}{s}\int_0^T \int_\Omega \frac{e^{-2s\eta}}{\varphi}(|\frac{\partial u}{\partial t}|^2 + \sum_{i,j=1}^N |\frac{\partial^2 u}{\partial x_i \partial x_j}|^2) dx dt \\ &+ s\lambda^2 \int_0^T \int_\Omega \varphi e^{-2s\eta} |\nabla u|^2 dx dt + s^3\lambda^4 \int_0^T \int_\Omega \varphi^3 e^{-2s\eta}|u|^2 dx dt \\ &\leq C(\int_0^T \int_\Omega e^{-2s\eta}|g|^2 dx dt + s^3\lambda^4 \int_0^T \int_\omega \varphi^3 e^{-2s\eta}|u|^2 dx dt). \end{aligned} \tag{6.4.46}$$

Proof. For $s > 0$ and $\lambda > 0$ we define

$$w(x,t) = e^{-s\eta(x,t)} u(x,t). \tag{6.4.47}$$

We can see that

$$w(x,0) = w(x,T) = 0. \tag{6.4.48}$$

We now compute in terms of w the operator

$$e^{-s\eta}(-\frac{\partial(e^{s\eta}w)}{\partial t} + L_0(e^{s\eta}w)) = e^{-s\eta}g. \tag{6.4.49}$$

We have

$$\frac{\partial(e^{s\eta}w)}{\partial t} = e^{s\eta}(\frac{\partial w}{\partial t} + s\frac{\partial \eta}{\partial t}w)$$

and because of (6.4.38)

$$\frac{\partial(e^{s\eta}w)}{\partial x_j} = e^{s\eta}(\frac{\partial w}{\partial x_j} - s\lambda\varphi\frac{\partial \psi}{\partial x_j}w)$$

so that

$$\frac{\partial}{\partial x_i}(a_{i,j}\frac{\partial(e^{s\eta}w)}{\partial x_j}) \tag{6.4.50}$$
$$= e^{s\eta}(\frac{\partial}{\partial x_i}(a_{i,j}\frac{\partial w}{\partial x_j}) - s\lambda\varphi a_{i,j}(\frac{\partial\psi}{\partial x_j}\frac{\partial w}{\partial x_i} + \frac{\partial\psi}{\partial x_i}\frac{\partial w}{\partial x_j})$$
$$- s\lambda\varphi\frac{\partial}{\partial x_i}(a_{i,j}\frac{\partial\psi}{\partial x_j})w - s\lambda^2\varphi a_{i,j}\frac{\partial\psi}{\partial x_j}\frac{\partial\psi}{\partial x_i}w + s^2\lambda^2\varphi^2 a_{i,j}\frac{\partial\psi}{\partial x_j}\frac{\partial\psi}{\partial x_i}w).$$

After minor modifications and using the symmetry of coefficients $a_{i,j}$ we then obtain

$$Pw = P_1w + P_2w = g_s \tag{6.4.51}$$

where

$$P_1w = -\frac{\partial w}{\partial t} + 2s\lambda\sum_{i,j=1}^{N}\varphi a_{i,j}\frac{\partial\psi}{\partial x_j}\frac{\partial w}{\partial x_i}$$
$$+2s\lambda^2\sum_{i,j=1}^{N}\varphi a_{i,j}\frac{\partial\psi}{\partial x_j}\frac{\partial\psi}{\partial x_i}w, \tag{6.4.52}$$

$$P_2w = -\sum_{i,j=1}^{N}\frac{\partial}{\partial x_i}(a_{i,j}\frac{\partial w}{\partial x_j})$$
$$-s^2\lambda^2\sum_{i,j=1}^{N}\varphi^2 a_{i,j}\frac{\partial\psi}{\partial x_j}\frac{\partial\psi}{\partial x_i}w - s\frac{\partial\eta}{\partial t}w, \tag{6.4.53}$$

$$g_s = e^{-s\eta}g + s\lambda^2\sum_{i,j=1}^{N}\varphi a_{i,j}\frac{\partial\psi}{\partial x_j}\frac{\partial\psi}{\partial x_i}w$$
$$-s\lambda\sum_{i,j=1}^{N}\varphi\frac{\partial}{\partial x_i}(a_{i,j}\frac{\partial\psi}{\partial x_j})w. \tag{6.4.54}$$

We now take the L^2-norm of each term in (6.4.51) and we obtain

$$\int_0^T\int_\Omega |P_1w|^2dxdt + \int_0^T\int_\Omega |P_2w|^2dxdt \tag{6.4.55}$$
$$+2\int_0^T\int_\Omega P_1wP_2wdxdt = \int_0^T\int_\Omega |g_s|^2dxdt.$$

We shall now compute the term

$$\int_0^T\int_\Omega P_1wP_2wdxdt$$

using (6.4.52) and (6.4.53). This computation will give 9 terms $I_{k,l}$. In the sequel, by C we mean various constants independent of s, λ and T as we want to keep track of the powers of s, λ and T involved. In order to organize the calculations we will give particular importance to terms of the order of

$$s\lambda^2 \int_0^T \int_\Omega \varphi |\nabla w|^2 dxdt$$

and

$$s^3\lambda^4 \int_0^T \int_\Omega \varphi^3 |w|^2 dxdt.$$

We take $s \geq 1$ and $\lambda \geq 1$ and we denote by

$$s^p \lambda^q A$$

all terms which can be bounded by

$$Cs^p\lambda^q \int_0^T \int_\Omega \varphi |\nabla w|^2, \quad p \leq 1,\ q \leq 2,\ p+q \leq 2$$

and by

$$s^p \lambda^q B$$

the terms which can be bounded by

$$Cs^p\lambda^q \int_0^T \int_\Omega \varphi^3 |w|^2 dxdt, \quad p \leq 3,\ q \leq 4,\ p+q \leq 6.$$

These terms will be neglectible as we will see later on. We have the following successive results,

$$\begin{aligned}
I_{1,1} &= \int_0^T \int_\Omega \frac{\partial w}{\partial t} \sum_{i,j=1}^N \frac{\partial}{\partial x_i}(a_{i,j}\frac{\partial w}{\partial x_j})dxdt \\
&= \sum_{i,j=1}^N \int_0^T \int_\Omega \frac{\partial}{\partial t}(\frac{\partial w}{\partial x_i})a_{i,j}\frac{\partial w}{\partial x_j}dxdt \\
&= \sum_{i,j=1}^N \frac{1}{2}\int_0^T \int_\Omega \frac{\partial}{\partial t}(a_{i,j}\frac{\partial w}{\partial x_j}\frac{\partial w}{\partial x_i})dxdt \\
&\quad - \sum_{i,j=1}^N \frac{1}{2}\int_0^T \int_\Omega (\frac{\partial a_{i,j}}{\partial t})\frac{\partial w}{\partial x_j}\frac{\partial w}{\partial x_i}dxdt \\
&= -\sum_{i,j=1}^N \frac{1}{2}\int_0^T \int_\Omega (\frac{\partial a_{i,j}}{\partial t})\frac{\partial w}{\partial x_j}\frac{\partial w}{\partial x_i}dxdt, \quad \text{because of (6.4.48).}
\end{aligned}$$

Then

$$
\begin{aligned}
I_{1,1} &= T^{2k}A, \qquad (6.4.56)\\
I_{1,2} &= s^2\lambda^2\int_0^T\int_\Omega \frac{\partial w}{\partial t}\sum_{i,j=1}^N \varphi^2 a_{i,j}\frac{\partial \psi}{\partial x_j}\frac{\partial \psi}{\partial x_i} w dxdt\\
&= \frac{s^2\lambda^2}{2}\sum_{i,j=1}^N\int_0^T\int_\Omega \varphi^2 a_{i,j}\frac{\partial \psi}{\partial x_j}\frac{\partial \psi}{\partial x_i}\frac{\partial}{\partial t}|w|^2 dxdt\\
&= -\frac{s^2\lambda^2}{2}\sum_{i,j=1}^N\int_0^T\int_\Omega (\frac{\partial a_{i,j}}{\partial t})\varphi^2 a_{i,j}\frac{\partial \psi}{\partial x_j}\frac{\partial \psi}{\partial x_i}|w|^2 dxdt\\
&\quad -s^2\lambda^2\sum_{i,j=1}^N\int_0^T\int_\Omega \varphi\frac{\partial \varphi}{\partial t} a_{i,j}\frac{\partial \psi}{\partial x_j}\frac{\partial \psi}{\partial x_i}|w|^2 dxdt.
\end{aligned}
$$

Because of (6.4.40) we have

$$
\begin{aligned}
I_{1,2} &= s^2\lambda^2(T^{2k}+T^{2k-1})B, \qquad (6.4.57)\\
I_{1,3} &= s\int_0^T\int_\Omega \frac{\partial w}{\partial t}\frac{\partial \eta}{\partial t} w = \frac{s}{2}\int_0^T\int_\Omega \frac{\partial \eta}{\partial t}\frac{\partial}{\partial t}(|w|^2)dxdt\\
&= -\frac{s}{2}\int_0^T\int_\Omega \frac{\partial^2 \eta}{\partial t^2}|w|^2 dxdt,
\end{aligned}
$$

and using (6.4.41)

$$
I_{1,3} = sT^{4k-2}B. \qquad (6.4.58)
$$

Before proceeding our calculations we have to notice that, if ν is the unit exterior normal on the boundary Γ, as ψ and w vanish on Γ we have for $x \in \Gamma$

$$
\nabla\psi(x,t) = (\nabla\psi\cdot\nu)\nu \quad \text{and} \quad \nabla w(x,t) = (\nabla w\cdot\nu)\nu,
$$

$$
\begin{aligned}
I_{2,1} &= -2s\lambda\sum_{i,j,k,l=1}^N\int_0^T\int_\Omega \varphi a_{i,j}\frac{\partial \psi}{\partial x_j}\frac{\partial w}{\partial x_i}\frac{\partial}{\partial x_k}(a_{k,l}\frac{\partial w}{\partial x_l})dxdt\\
&= -2s\lambda\sum_{i,j,k,l=1}^N\int_0^T\int_\Gamma \varphi(\nabla\psi\cdot\nu)|\nabla w.\nu|^2 a_{i,j}\nu_i\nu_j a_{k,l}\nu_k\nu_l d\gamma dt\\
&\quad +2s\lambda^2\sum_{i,j,k,l=1}^N\int_0^T\int_\Omega \varphi(a_{i,j}\frac{\partial \psi}{\partial x_j}\frac{\partial w}{\partial x_i})(a_{k,l}\frac{\partial \psi}{\partial x_l}\frac{\partial w}{\partial x_k})dxdt\\
&\quad +2s\lambda\sum_{i,j,k,l=1}^N\int_0^T\int_\Omega \varphi\frac{\partial}{\partial x_k}(a_{i,j}\frac{\partial \psi}{\partial x_j})a_{k,l}\frac{\partial w}{\partial x_i}\frac{\partial w}{\partial x_l}dxdt
\end{aligned}
$$

$$+2s\lambda \sum_{i,j,k,l=1}^{N} \int_0^T \int_\Omega \varphi a_{i,j} \frac{\partial \psi}{\partial x_j} a_{k,l} \frac{\partial w}{\partial x_l} \frac{\partial^2 w}{\partial x_k \partial x_i} dxdt$$

$$= -s\lambda \sum_{i,j,k,l=1}^{N} \int_0^T \int_\Gamma \varphi(\nabla\psi \cdot \nu)|\nabla w \cdot \nu|^2 a_{i,j}\nu_i\nu_j a_{k,l}\nu_k\nu_l d\gamma dt$$

$$+2s\lambda^2 \sum_{i,j,k,l=1}^{N} \int_0^T \int_\Omega \varphi(a_{i,j} \frac{\partial \psi}{\partial x_j} \frac{\partial w}{\partial x_i})(a_{k,l} \frac{\partial \psi}{\partial x_l} \frac{\partial w}{\partial x_k}) dxdt$$

$$+2s\lambda \sum_{i,j,k,l=1}^{N} \int_0^T \int_\Omega \varphi \frac{\partial}{\partial x_k}(a_{i,j} \frac{\partial \psi}{\partial x_j}) a_{k,l} \frac{\partial w}{\partial x_i} \frac{\partial w}{\partial x_l} dxdt$$

$$-s\lambda^2 \sum_{i,j,k,l=1}^{N} \int_0^T \int_\Omega \varphi a_{i,j} \frac{\partial \psi}{\partial x_j} \frac{\partial \psi}{\partial x_i} a_{k,l} \frac{\partial w}{\partial x_l} \frac{\partial w}{\partial x_k} dxdt$$

$$-s\lambda \sum_{i,j,k,l=1}^{N} \int_0^T \int_\Omega \varphi \frac{\partial}{\partial x_i}(a_{i,j} \frac{\partial \psi}{\partial x_j}) a_{k,l} \frac{\partial w}{\partial x_l} \frac{\partial w}{\partial x_k} dxdt$$

$$-s\lambda \sum_{i,j,k,l=1}^{N} \int_0^T \int_\Omega \varphi a_{i,j} \frac{\partial \psi}{\partial x_j} \frac{\partial a_{k,l}}{\partial x_i} \frac{\partial w}{\partial x_l} \frac{\partial w}{\partial x_k} dxdt.$$

Therefore we have

$$\begin{aligned} I_{2,1} = & -s\lambda \sum_{i,j,k,l=1}^{N} \int_0^T \int_\Gamma \varphi(\nabla\psi \cdot \nu)|\nabla w \cdot \nu|^2 a_{i,j}\nu_i\nu_j a_{k,l}\nu_k\nu_l d\gamma dt \\ & +2s\lambda^2 \sum_{i,j,k,l=1}^{N} \int_0^T \int_\Omega \varphi(a_{i,j} \frac{\partial \psi}{\partial x_j} \frac{\partial w}{\partial x_i})(a_{k,l} \frac{\partial \psi}{\partial x_l} \frac{\partial w}{\partial x_k}) dxdt \\ & -s\lambda^2 \sum_{i,j,k,l=1}^{N} \int_0^T \int_\Omega \varphi a_{i,j} \frac{\partial \psi}{\partial x_j} \frac{\partial \psi}{\partial x_i} a_{k,l} \frac{\partial w}{\partial x_l} \frac{\partial w}{\partial x_k} dxdt \\ & +s\lambda A, \end{aligned} \tag{6.4.59}$$

$$\begin{aligned} I_{2,2} = & -2s^3\lambda^3 \sum_{i,j,k,l=1}^{N} \int_0^T \int_\Omega \varphi^3 a_{i,j} \frac{\partial \psi}{\partial x_j} \frac{\partial w}{\partial x_i} a_{k,l} \frac{\partial \psi}{\partial x_l} \frac{\partial \psi}{\partial x_k} w dxdt \\ = & -s^3\lambda^3 \sum_{i,j,k,l=1}^{N} \int_0^T \int_\Omega \varphi^3 a_{i,j} \frac{\partial \psi}{\partial x_j} \frac{\partial}{\partial x_i}(|w|^2) a_{k,l} \frac{\partial \psi}{\partial x_l} \frac{\partial \psi}{\partial x_k} dxdt \\ = & \, 3s^3\lambda^4 \sum_{i,j,k,l=1}^{N} \int_0^T \int_\Omega \varphi^3 a_{i,j} \frac{\partial \psi}{\partial x_j} \frac{\partial \psi}{\partial x_i} a_{k,l} \frac{\partial \psi}{\partial x_l} \frac{\partial \psi}{\partial x_k} |w|^2 dxdt \end{aligned}$$

$$+s^3\lambda^3 \sum_{i,j,k,l=1}^{N} \int_0^T \int_\Omega \varphi^3 \frac{\partial}{\partial x_i}(a_{i,j}\frac{\partial \psi}{\partial x_j}) a_{k,l}\frac{\partial \psi}{\partial x_l}\frac{\partial \psi}{\partial x_k}|w|^2 dxdt$$

$$+s^3\lambda^3 \sum_{i,j,k,l=1}^{N} \int_0^T \int_\Omega \varphi^3 a_{i,j}\frac{\partial \psi}{\partial x_j}\frac{\partial}{\partial x_i}(a_{k,l}\frac{\partial \psi}{\partial x_l}\frac{\partial \psi}{\partial x_k})|w|^2 dxdt,$$

so that

$$I_{2,2} = 3s^3\lambda^4 \sum_{i,j,k,l=1}^{N} \int_0^T \int_\Omega \varphi^3 a_{i,j}\frac{\partial \psi}{\partial x_j}\frac{\partial \psi}{\partial x_i} a_{k,l}\frac{\partial \psi}{\partial x_l}\frac{\partial \psi}{\partial x_k}|w|^2 dxdt \quad (6.4.60)$$
$$+s^3\lambda^3 B,$$

$$I_{2,3} = -2s^2\lambda \sum_{i,j=1}^{N} \int_0^T \int_\Omega \varphi a_{i,j}\frac{\partial \psi}{\partial x_j}\frac{\partial w}{\partial x_i}\frac{\partial \eta}{\partial t} w dxdt$$
$$= -s^2\lambda \sum_{i,j=1}^{N} \int_0^T \int_\Omega \varphi \frac{\partial \eta}{\partial t} a_{i,j}\frac{\partial \psi}{\partial x_j}\frac{\partial}{\partial x_i}|w|^2 dxdt$$
$$= s^2\lambda^2 \sum_{i,j=1}^{N} \int_0^T \int_\Omega \varphi \frac{\partial \eta}{\partial t} a_{i,j}\frac{\partial \psi}{\partial x_j}\frac{\partial \psi}{\partial x_i}|w|^2 dxdt$$
$$-s^2\lambda^2 \sum_{i,j=1}^{N} \int_0^T \int_\Omega \varphi \frac{\partial \varphi}{\partial t} a_{i,j}\frac{\partial \psi}{\partial x_j}\frac{\partial \psi}{\partial x_i}|w|^2 dxdt$$
$$+s^2\lambda \sum_{i,j=1}^{N} \int_0^T \int_\Omega \varphi \frac{\partial \eta}{\partial t}\frac{\partial}{\partial x_i}(a_{i,j}\frac{\partial \psi}{\partial x_j})|w|^2 dxdt,$$

so that

$$I_{2,3} = s^2\lambda^2 T^{2k-1} B, \quad (6.4.61)$$

$$I_{3,1} = -2s\lambda^2 \sum_{i,j,k,l=1}^{N} \int_0^T \int_\Omega \varphi a_{i,j}\frac{\partial \psi}{\partial x_j}\frac{\partial \psi}{\partial x_i} w \frac{\partial}{\partial x_k}(a_{k,l}\frac{\partial w}{\partial x_l}) dxdt$$
$$= 2s\lambda^3 \sum_{i,j,k,l=1}^{N} \int_0^T \int_\Omega \varphi a_{i,j}\frac{\partial \psi}{\partial x_j}\frac{\partial \psi}{\partial x_i}\frac{\partial \psi}{\partial x_k} a_{k,l}\frac{\partial w}{\partial x_l} w dxdt$$
$$+2s\lambda^2 \sum_{i,j,k,l=1}^{N} \int_0^T \int_\Omega \varphi \frac{\partial}{\partial x_k}(a_{i,j}\frac{\partial \psi}{\partial x_j}\frac{\partial \psi}{\partial x_i}) a_{k,l}\frac{\partial w}{\partial x_l} w dxdt$$
$$+2s\lambda^2 \sum_{i,j,k,l=1}^{N} \int_0^T \int_\Omega \varphi a_{i,j}\frac{\partial \psi}{\partial x_j}\frac{\partial \psi}{\partial x_i} a_{k,l}\frac{\partial w}{\partial x_l}\frac{\partial w}{\partial x_k} dxdt$$

and we have

$$I_{3,1} = 2s\lambda^2 \sum_{i,j,k,l=1}^{N} \int_0^T \int_\Omega \varphi a_{i,j} \frac{\partial \psi}{\partial x_j} \frac{\partial \psi}{\partial x_i} a_{k,l} \frac{\partial w}{\partial x_l} \frac{\partial w}{\partial x_k} dxdt \qquad (6.4.62)$$
$$+ s\lambda^3 T^{2k} \sqrt{AB}.$$

Now we directly get

$$I_{3,2} = -2s^3\lambda^4 \sum_{i,j,k,l=1}^{N} \int_0^T \int_\Omega \varphi^3 (a_{i,j} \frac{\partial \psi}{\partial x_j} \frac{\partial \psi}{\partial x_i})(a_{k,l} \frac{\partial \psi}{\partial x_l} \frac{\partial \psi}{\partial x_k}) |w|^2 dxdt, \qquad (6.4.63)$$

and

$$I_{3,3} = -2s^2\lambda^2 \int_0^T \int_\Omega \varphi \frac{\partial \eta}{\partial t} (a_{i,j} \frac{\partial \psi}{\partial x_j} \frac{\partial \psi}{\partial x_i}) |w|^2 dxdt = s^2 \lambda^2 T^{2k-1} B. \quad (6.4.64)$$

Grouping all different terms $I_{k,l}$ we obtain

$$\begin{aligned} & 2\int_0^T \int_\Omega P_1 w P_2 w dxdt \\ &= (s\lambda + \lambda^2 T^{2k}) A + (s^3\lambda^3 + s^2\lambda^4 T^{2k} + s^2\lambda^2 T^{2k-1} + sT^{4k-2}) B \\ &+ 2s\lambda^2 \sum_{i,j,k,l=1}^{N} \int_0^T \int_\Omega \varphi a_{i,j} \frac{\partial \psi}{\partial x_j} \frac{\partial \psi}{\partial x_i} a_{k,l} \frac{\partial w}{\partial x_l} \frac{\partial w}{\partial x_k} dxdt \\ &+ 2s^3\lambda^4 \sum_{i,j,k,l=1}^{N} \int_0^T \int_\Omega \varphi^3 (a_{i,j} \frac{\partial \psi}{\partial x_j} \frac{\partial \psi}{\partial x_i})(a_{k,l} \frac{\partial \psi}{\partial x_l} \frac{\partial \psi}{\partial x_k}) |w|^2 dxdt \\ &- 2s\lambda \sum_{i,j,k,l=1}^{N} \int_0^T \int_\Gamma \varphi (\nabla \psi \cdot \nu) |\nabla w \cdot \nu|^2 a_{i,j} \nu_i \nu_j a_{k,l} \nu_k \nu_l d\gamma dt \\ &+ 4s\lambda^2 \sum_{i,j,k,l=1}^{N} \int_0^T \int_\Omega \varphi (a_{i,j} \frac{\partial \psi}{\partial x_j} \frac{\partial w}{\partial x_i})(a_{k,l} \frac{\partial \psi}{\partial x_l} \frac{\partial w}{\partial x_k}) dxdt. \end{aligned}$$

We know that $\nabla \psi \cdot \nu < 0$ on Γ so that the boundary integral (with the $-$ sign) is positive. Using (6.1.4) we then have, writing

$$A(s,\lambda,T) = (s\lambda + \lambda^2 T^{2k}) A$$

and

$$B(s,\lambda,T) = (s^3\lambda^3 + s^2\lambda^4 T^{2k} + s^2\lambda^2 T^{2k-1} + sT^{4k-2}) B,$$

$$2\int_0^T \int_\Omega P_1 w P_2 w dxdt \geq A(s,\lambda,T) + B(s,\lambda,T) \qquad (6.4.65)$$
$$+ 2s\lambda^2\beta^2 \int_0^T \int_\Omega \varphi |\nabla \psi|^2 |\nabla w|^2 dxdt + 2s^3\lambda^4\beta^4 \int_0^T \int_\Omega \varphi^3 |\nabla \psi|^4 |w|^2 dxdt.$$

Because $|\nabla\psi| \neq 0$ on $\overline{\Omega - \omega_0}$ (see Lemma 6.4.1), there exists $\delta > 0$ such that

$$\beta|\nabla\psi| \geq \delta \quad \text{on } \overline{\Omega - \omega_0},$$

so that

$$\begin{aligned}
&2\int_0^T\int_\Omega P_1wP_2w\,dxdt + 2s\lambda^2\delta^2\int_0^T\int_{\omega_0}\varphi|\nabla w|^2dxdt \\
&+2s^3\lambda^4\delta^4\int_0^T\int_{\omega_0}\varphi^3|w|^2dxdt \\
\geq\ & 2s\lambda^2\delta^2\int_0^T\int_\Omega\varphi|\nabla w|^2dxdt + 2s^3\lambda^4\delta^4\int_0^T\int_\Omega\varphi^3|w|^2dxdt \\
&+A(s,\lambda,T) + B(s,\lambda,T).
\end{aligned}$$

We also have

$$\int_0^T\int_\Omega|g_s|^2dxdt \leq \int_0^T\int_\Omega e^{-2s\eta}|g|^2dxdt + B(s,\lambda,T),$$

so that

$$\begin{aligned}
&\int_0^T\int_\Omega|P_1w|^2dxdt + \int_0^T\int_\Omega|P_2w|^2dxdt \\
&+2s\lambda^2\delta^2\int_0^T\int_\Omega\varphi|\nabla w|^2dxdt + 2s^3\lambda^4\delta^4\int_0^T\int_\Omega\varphi^3|w|^2dxdt \\
\leq\ & \int_0^T\int_\Omega e^{-2s\eta}|g|^2dxdt + 2s\lambda^2\delta^2\int_0^T\int_{\omega_0}\varphi|\nabla w|^2dxdt \\
&+2s^3\lambda^4\delta^4\int_0^T\int_{\omega_0}\varphi^3|w|^2dxdt + A(s,\lambda,T) + B(s,\lambda,T).
\end{aligned}$$

Because of the form of $A(s,\lambda,T)$ and $B(s,\lambda,T)$, we can eliminate them by choosing s and λ sufficiently large, say $s \geq (T^{2k}+T^{2k-1})s_0$ and $\lambda \geq \lambda_0$ where s_0 and λ_0 are independent of T. Therefore we have for $s \geq (T^{2k} + T^{2k-1})s_0$ and $\lambda \geq \lambda_0$

$$\begin{aligned}
&\int_0^T\int_\Omega|P_1w|^2dxdt + \int_0^T\int_\Omega|P_2w|^2dxdt \qquad (6.4.66)\\
&+s\lambda^2\delta^2\int_0^T\int_\Omega\varphi|\nabla w|^2dxdt + s^3\lambda^4\delta^4\int_0^T\int_\Omega\varphi^3|w|^2dxdt \\
\leq\ & \int_0^T\int_\Omega e^{-2s\eta}|g|^2dxdt + 2s\lambda^2\delta^2\int_0^T\int_{\omega_0}\varphi|\nabla w|^2dxdt \\
&+2s^3\lambda^4\delta^4\int_0^T\int_{\omega_0}\varphi^3|w|^2dxdt.
\end{aligned}$$

Now we want to get rid of the term $2s\lambda^2\delta^2 \int_0^T \int_{\omega_0} \varphi |\nabla w|^2 dxdt$ in the right-hand side of (6.4.66). To this aim let us introduce a cut-off function θ such that

$$\theta \in \mathcal{D}(\omega), \quad 0 \le \theta \le 1, \quad \theta(x) = 1 \quad \forall x \in \omega_0. \tag{6.4.67}$$

We multiply $P_2 w$ by $\varphi\theta^2 w$ which gives

$$\begin{aligned}
&\int_0^T \int_\Omega P_2 w \varphi \theta^2 w dxdt \\
&= -s \int_0^T \int_\Omega \frac{\partial \eta}{\partial t} w^2 \varphi \theta^2 dxdt + \int_0^T \int_\Omega \sum_{i,j=1}^N a_{i,j} \frac{\partial w}{\partial x_j} \frac{\partial w}{\partial x_i} \varphi \theta^2 dxdt \\
&+ \int_0^T \int_\Omega 2 \sum_{i,j=1}^N a_{i,j} \frac{\partial w}{\partial x_j} \frac{\partial \theta}{\partial x_i} w \varphi \theta dxdt \\
&+ \lambda \int_0^T \int_\Omega \sum_{i,j=1}^N a_{i,j} \frac{\partial w}{\partial x_j} \frac{\partial \psi}{\partial x_i} w \varphi \theta^2 dxdt \\
&- s^2\lambda^2 \int_0^T \int_\Omega \sum_{i,j=1}^N \varphi^3 a_{i,j} \frac{\partial \psi}{\partial x_j} \frac{\partial \psi}{\partial x_i} w^2 \theta^2 dxdt.
\end{aligned}$$

Therefore, using (6.1.4) we have

$$\begin{aligned}
&\int_0^T \int_\omega \varphi \theta^2 |\nabla w|^2 dxdt \\
&\le C\Big(\int_0^T \int_\Omega P_2 w \varphi \theta^2 w dxdt + \lambda \int_0^T \int_\omega \varphi^{\frac{1}{2}} \theta |\nabla w| \varphi^{\frac{1}{2}} w dxdt \\
&+ (s^2\lambda^2 + sT^{2k-1}) \int_0^T \int_\omega \varphi^3 w^2 dxdt\Big).
\end{aligned}$$

We then obtain for $s \ge (T^{2k} + T^{2k-1})s_0$ and $\lambda \ge \lambda_0$,

$$\begin{aligned}
&2s\lambda^2\delta^2 \int_0^T \int_{\omega_0} \varphi |\nabla w|^2 dxdt \\
&\le \frac{1}{2} \int_0^T \int_\Omega |P_2 w|^2 dxdt + Cs^3\lambda^4 \int_0^T \int_\omega \varphi^3 w^2 dxdt.
\end{aligned}$$

From (6.4.66) this gives

$$\begin{aligned}
&\int_0^T \int_\Omega |P_1 w|^2 dxdt + \int_0^T \int_\Omega |P_2 w|^2 dxdt \\
&+ s\lambda^2 \int_0^T \int_\Omega \varphi |\nabla w|^2 dxdt + s^3\lambda^4 \int_0^T \int_\Omega \varphi^3 |w|^2 dxdt \\
&\le C\Big(\int_0^T \int_\Omega e^{-2s\eta} |g|^2 dxdt + s^3\lambda^4 \int_0^T \int_\omega \varphi^3 |w|^2 dxdt\Big).
\end{aligned} \tag{6.4.68}$$

We want to give this inequality in terms of u instead of w. We know that $w = e^{-s\eta}u$. Therefore we have

$$\int_0^T \int_\Omega \varphi^3 |w|^2 dxdt = \int_0^T \int_\Omega \varphi^3 e^{-2s\eta} |u|^2 dxdt$$

and because of

$$\nabla u = e^{s\eta}(\nabla w - s\lambda\varphi\nabla\psi w),$$

$$\int_0^T \int_\Omega \varphi e^{-2s\eta} |\nabla u|^2 dxdt \leq C(\int_0^T \int_\Omega \varphi |\nabla w|^2 dxdt + s^2\lambda^2 \int_0^T \int_\Omega \varphi^3 |w|^2 dxdt).$$

This immediately gives one part of (6.4.46). In order to obtain the complete inequality (6.4.46), we use the explicit form of $P_1 w$ and $P_2 w$ which from (6.4.68) give for $s \geq (T^{2k} + T^{2k-1})s_0$ and $\lambda \geq \lambda_0$

$$\frac{1}{s}\int_0^T \int_\Omega \frac{1}{\varphi} |\frac{\partial w}{\partial t}|^2 \leq C(\int_0^T \int_\Omega e^{-2s\eta}|g|^2 dxdt + s^3\lambda^4 \int_0^T \int_\omega \varphi^3 |w|^2 dxdt),$$

and

$$\frac{1}{s}\int_0^T \int_\Omega \frac{1}{\varphi} |\sum_{i,j=1}^N \frac{\partial}{\partial x_i}(a_{i,j}\frac{\partial w}{\partial x_j})|^2 dxdt$$
$$\leq C(\int_0^T \int_\Omega e^{-2s\eta}|g|^2 dxdt + s^3\lambda^4 \int_0^T \int_\omega \varphi^3 |w|^2 dxdt).$$

Developing the expression $\sum_{i,j=1}^N \frac{\partial}{\partial x_i}(a_{i,j}\frac{\partial}{\partial x_j}(\frac{w}{\sqrt{\varphi}}))$ and using again estimate (6.4.68), $s \geq (T^{2k} + T^{2k-1})s_0$ and $\lambda \geq \lambda_0$, we obtain

$$\frac{1}{s}\int_0^T \int_\Omega |\sum_{i,j=1}^N \frac{\partial}{\partial x_i}(a_{i,j}\frac{\partial}{\partial x_j}(\frac{w}{\sqrt{\varphi}}))|^2 dxdt$$
$$\leq C(\int_0^T \int_\Omega e^{-2s\eta}|g|^2 dxdt + s^3\lambda^4 \int_0^T \int_\omega \varphi^3 |w|^2 dxdt).$$

Using elliptic regularity results in Ω, we obtain from this last inequality

$$\frac{1}{s}\int_0^T \int_\Omega \sum_{i,j=1}^N |\frac{\partial^2}{\partial x_i \partial x_j}(\frac{w}{\sqrt{\varphi}})|^2 dxdt$$
$$\leq C(\int_0^T \int_\Omega e^{-2s\eta}|g|^2 dxdt + s^3\lambda^4 \int_0^T \int_\omega \varphi^3 |w|^2 dxdt).$$

Using the development of $\frac{\partial^2}{\partial x_i \partial x_j}(\frac{w}{\sqrt{\varphi}})$ together with (6.4.68), $s \geq (T^{2k} + T^{2k-1})s_0$ and $\lambda \geq \lambda_0$ gives now

$$\frac{1}{s}\int_0^T \int_\Omega \frac{1}{\varphi}(\sum_{i,j=1}^N |\frac{\partial^2 w}{\partial x_i \partial x_j}|^2)dxdt$$
$$\leq C(\int_0^T \int_\Omega e^{-2s\eta}|g|^2 dxdt + s^3\lambda^4 \int_0^T \int_\omega \varphi^3 |w|^2 dxdt).$$

We know that

$$\frac{\partial u}{\partial t} = \frac{\partial (e^{s\eta} w)}{\partial t} = e^{s\eta}(\frac{\partial w}{\partial t} + s\frac{\partial \eta}{\partial t}w)$$

and

$$\begin{aligned}\frac{\partial^2 u}{\partial x_i \partial x_j} &= \frac{\partial^2 (e^{s\eta} w)}{\partial x_i \partial x_j} \\ &= e^{s\eta}(\frac{\partial^2 w}{\partial x_i \partial x_j} - s\lambda\varphi(\frac{\partial \psi}{\partial x_j}\frac{\partial w}{\partial x_i} + \frac{\partial \psi}{\partial x_i}\frac{\partial w}{\partial x_j}) \\ &\quad -s\lambda\varphi\frac{\partial^2 \psi}{\partial x_i \partial x_j}w - s\lambda^2\varphi\frac{\partial \psi}{\partial x_j}\frac{\partial \psi}{\partial x_i}w + s^2\lambda^2\varphi^2\frac{\partial \psi}{\partial x_j}\frac{\partial \psi}{\partial x_i}w).\end{aligned}$$

We then obtain

$$\frac{1}{s}\int_0^T \int_\Omega \frac{e^{-2s\eta}}{\varphi}(|\frac{\partial u}{\partial t}|^2 + \sum_{i,j=1}^N |\frac{\partial^2 u}{\partial x_i \partial x_j}|^2)dxdt$$
$$\leq C(\int_0^T \int_\Omega e^{-2s\eta}|g|^2 dxdt + s^3\lambda^4 \int_0^T \int_\omega \varphi^3 e^{-2s\eta}|u|^2 dxdt),$$

with C independent of s, λ and T and this finishes the proof of Theorem 6.4.3.

6.4.3. *Case of a General Diffusion-Convection Operator*

For the moment we have proved a Carleman inequality for a solution of (6.4.43) where only operator L_0 is considered. If we take the general case of operator L^* like in (6.2.24) we can obtain a similar inequality. Let us consider the general problem

$$-\frac{\partial u}{\partial t} + L^* u = h \quad \text{in } \Omega \times (0,T), \tag{6.4.69}$$
$$u = 0 \quad \text{on } \Gamma \times (0,T), \tag{6.4.70}$$
$$u(T) = u_0 \quad \text{in } \Omega. \tag{6.4.71}$$

We write

$$|a_0|_\infty = |a_0|_{L^\infty(\Omega)} \quad \text{and} \quad |b|_\infty = \max_{i=1,\cdots,N} |b_i|_{L^\infty(\Omega)}, \tag{6.4.72}$$

and we have the following result corresponding to Theorem 6.4.3.

Theorem 6.4.4. *Let s_0, λ_0 be defined as in Theorem 6.4.3. There exists a constant $C > 0$ depending on Ω, ω_0, ψ, on β defined in (6.1.4) and on coefficients $a_{i,j}$ such that, if*

$$s_1 = C(1 + |a_0|_\infty^{\frac{2}{3}} + |b|_\infty^2), \tag{6.4.73}$$

for every $s > T^{2k}s_1 + T^{2k-1}s_0$, for every $\lambda > \lambda_0$ and for every solution of (6.4.69) we have

$$\frac{1}{s}\int_0^T\int_\Omega \frac{e^{-2s\eta}}{\varphi}(|\frac{\partial u}{\partial t}|^2 + \sum_{i,j=1}^N |\frac{\partial^2 u}{\partial x_i \partial x_j}|^2)dxdt \tag{6.4.74}$$
$$+s\lambda^2\int_0^T\int_\Omega \varphi e^{-2s\eta}|\nabla u|^2 dxdt + s^3\lambda^4\int_0^T\int_\Omega \varphi^3 e^{-2s\eta}|u|^2 dxdt$$
$$\leq C(\int_0^T\int_\Omega e^{-2s\eta}|h|^2 dxdt + s^3\lambda^4\int_0^T\int_\omega \varphi^3 e^{-2s\eta}|u|^2 dxdt).$$

Proof. It is a simple consequence of the previous result. In fact let us define

$$g = h + \sum_{i=1}^N b_i\frac{\partial u}{\partial x_i} - a_0 u. \tag{6.4.75}$$

Then the solution of (6.4.69) is solution of (6.4.43) with g defined by (6.4.75). We can then apply Theorem 6.4.3 and obtain (6.4.46) for $s \geq (T^{2k} + T^{2k-1})s_0$ and $\lambda \geq \lambda_0$.

On the other hand the value of g gives immediately

$$\int_0^T\int_\Omega e^{-2s\eta}|g|^2 dxdt \leq C(\int_0^T\int_\Omega e^{-2s\eta}|h|^2 dxdt$$
$$+|b|_\infty^2 T^{2k}\int_0^T\int_\Omega \varphi e^{-2s\eta}|\nabla u|^2 dxdt + |a_0|_\infty^2 T^{6k}\int_0^T\int_\Omega \varphi^3 e^{-2s\eta}|u|^2 dxdt),$$

and therefore

$$\frac{1}{s}\int_0^T\int_\Omega \frac{e^{-2s\eta}}{\varphi}(|\frac{\partial u}{\partial t}|^2 + \sum_{i,j=1}^N |\frac{\partial^2 u}{\partial x_i \partial x_j}|^2)dxdt$$
$$+s\lambda^2\int_0^T\int_\Omega \varphi e^{-2s\eta}|\nabla u|^2 dxdt + s^3\lambda^4\int_0^T\int_\Omega \varphi^3 e^{-2s\eta}|u|^2 dxdt$$

$$\leq C\Big(\int_0^T\int_\Omega e^{-2s\eta}|h|^2dxdt + s^3\lambda^4\int_0^T\int_\omega \varphi^3e^{-2s\eta}|u|^2dxdt\Big)$$
$$+C|b|_\infty^2T^{2k}\int_0^T\int_\Omega \varphi e^{-2s\eta}|\nabla u|^2dxdt + C|a_0|_\infty^2T^{6k}\int_0^T\int_\Omega \varphi^3e^{-2s\eta}|u|^2dxdt.$$

Therefore by choosing

$$s_1 = s_0 + C^{\frac{1}{3}}|a_0|_\infty^{\frac{2}{3}} + C|b|_\infty^2, \tag{6.4.76}$$

taking $s \geq T^{2k}s_1 + T^{2k-1}s_0$ and $\lambda \geq \lambda_0$, we can absorb both of the two last terms in the right-hand side of the previous inequality by the left hand side. This gives immediately the result of Theorem 6.4.4.

6.5. *Observability Inequality*

In order to complete the proof of Theorem 6.1.2 which gives the result of exact controllability to trajectories, we need to prove the observability inequality (6.3.32) with a suitable weight function ρ.

Directly from (6.4.74) we have for $s \geq T^{2k}s_1 + T^{2k-1}s_0$ and $\lambda \geq \lambda_0$,

$$s^3\lambda^4\int_0^T\int_\Omega \varphi^3e^{-2s\eta}|u|^2dxdt \leq C\Big(\int_0^T\int_\Omega e^{-2s\eta}|h|^2dxdt \tag{6.5.77}$$
$$+s^3\lambda^4\int_0^T\int_\omega \varphi^3e^{-2s\eta}|u|^2dxdt\Big).$$

Now we can fix parameters s and λ in the admissible range and for example we take, in order to simplify notations, for C large enough,

$$s = CT^{2k}\Big(1 + |a_0|_\infty^{\frac{2}{3}} + |b|_\infty^2 + \frac{1}{T}\Big), \quad \lambda = \lambda_0.$$

We shall now only keep track of the dependence of the constants on T, $|a_{0|_\infty}$ and $|b|_\infty$.

On a time interval $(\frac{T}{4}, \frac{3T}{4})$, we have

$$\eta \leq \frac{C}{T^{2k}}, \quad \text{so that } e^{-2s\eta} \geq e^{-C(1+|a_0|_\infty^{\frac{2}{3}}+|b|_\infty^2+\frac{1}{T})}$$

and

$$\varphi \geq \frac{C}{T^{2k}}.$$

From the last inequality (6.5.77), we have

$$\int_{\frac{T}{4}}^{\frac{3T}{4}}\int_\Omega |u|^2dxdt \leq C(T,|a_0|_\infty,|b|_\infty)\int_0^T\int_\Omega e^{-2s\eta}|h|^2dxdt \tag{6.5.78}$$
$$+C(T,|a_0|_\infty,|b|_\infty)T^{6k}\int_0^T\int_\omega \varphi^3e^{-2s\eta}|u|^2dxdt,$$

where

$$C(T, |a_0|_\infty, |b|_\infty) = Ce^{C(1+|a_0|_\infty^{\frac{2}{3}}+|b|_\infty^2+\frac{1}{T})}. \tag{6.5.79}$$

Let us consider a cut-off function of time t θ such that

$$\theta \in C^\infty([0,T]), \quad 0 \leq \theta(t) \leq 1, \quad |\frac{d\theta}{dt}(t)| \leq \frac{C}{T}, \quad \forall t \in [0,T],$$

$$\theta(t) = 1, \quad \forall t \in [0, \frac{T}{2}], \quad \theta(t) = 0, \quad \forall t \in [\frac{3T}{4}, T].$$

We define

$$\tilde{u}(x,t) = \theta(t)u(x,t).$$

Then $\tilde{u}$ satisfies the following equation

$$-\frac{\partial \tilde{u}}{\partial t} + L^*\tilde{u} = \theta h - \frac{d\theta}{dt}u \quad \text{in } \Omega \times (0,T), \tag{6.5.80}$$

$$\tilde{u} = 0 \quad \text{on } \Gamma \times (0,T), \tag{6.5.81}$$

$$\tilde{u}(T) = 0 \quad \text{in } \Omega. \tag{6.5.82}$$

We now use a classical energy estimate for $\tilde{u}$. We have to notice that $\tilde{u}(x,t) = u(x,t)$ on $[0, \frac{T}{2}]$ and that $\frac{d\theta}{dt} = 0$ on $[0, \frac{T}{2}] \cup [\frac{3T}{4}, T]$. Multiplying the previous equation by $\tilde{u}$ and integrating over (t,T) for $t \in (0,T)$, we easily find that

$$|\tilde{u}(t)|^2_{L^2(\Omega)} + \beta \int_t^T \int_\Omega |\nabla \tilde{u}|^2 dxdt \leq C \int_t^{\frac{3T}{4}} \int_\Omega |h|^2 dxdt$$

$$+C(|a_0|_\infty + |b|^2_\infty) \int_t^T \int_\Omega |\tilde{u}|^2 dxdt + \frac{C}{T^2} \int_{\frac{T}{2}}^{\frac{3T}{4}} \int_\Omega |u|^2 dxdt.$$

From Gronwall's Lemma, we therefore have

$$|\tilde{u}(t)|^2_{L^2(\Omega)} \leq Ce^{C(|a_0|_\infty+|b|^2_\infty)T} (\int_0^{\frac{3T}{4}} \int_\Omega |h|^2 dxdt + \frac{1}{T^2} \int_{\frac{T}{2}}^{\frac{3T}{4}} \int_\Omega |u|^2 dxdt).$$

Using (6.5.78) we obtain in particular

Proposition 6.5.1. *For every solution u of (6.4.69), we have the following observability inequality.*

$$|u(0)|^2_{L^2(\Omega)} \leq Ce^{C(|a_0|_\infty+|b|^2_\infty)T} (\int_0^{\frac{3T}{4}} \int_\Omega |h|^2 dxdt \tag{6.5.83}$$

$$+C(T, |a_0|_\infty, |b|_\infty)(\int_0^T \int_\Omega e^{-2s\eta} |h|^2 dxdt$$

$$+T^{6k-2} \int_0^T \int_\omega \varphi^3 e^{-2s\eta} |u|^2 dxdt)),$$

where $C(T, |a_0|_\infty, |b|_\infty)$ is defined by (6.5.79).

Applying (6.5.83) to ξ_ϵ which is solution to (6.2.24) or to (6.4.69) with $h = 0$, we easily obtain one part of the desired observability inequality (6.3.32), namely

$$|\xi_\epsilon(0)|^2_{L^2(\Omega)} \leq CT^{6k-2}e^{K(T,|a_0|_\infty,|b|_\infty)}\int_0^T\int_\omega \varphi^3 e^{-2s\eta}|\xi_\epsilon|^2 dxdt, \quad (6.5.84)$$

where

$$K(T,|a_0|_\infty,|b|_\infty) = C(1+|a_0|_\infty^{\frac{2}{3}} + |b|^2_\infty + T|a_0|_\infty + T|b|^2_\infty + \frac{1}{T}). \quad (6.5.85)$$

This completes the proof of Theorem 6.1.2 in the case $g^0 = 0$ as it is announced. Here we have been keeping track of the dependence of the constants on T, $|a_0|_\infty$ and $|b|_\infty$ in order to treat nonlinear problems later on.

Let us now for simplicity forget about this dependence and define

$$\tilde{\eta}(x,t) = \begin{cases} \eta(x,t), & \text{if } t \in [\frac{T}{2}, T], \\ \eta(x,\frac{T}{2}), & \text{if } t \in [0, \frac{T}{2}] \end{cases} \quad (6.5.86)$$

and

$$\tilde{\varphi}(x,t) = \begin{cases} \varphi(x,t), & \text{if } t \in [\frac{T}{2}, T], \\ \varphi(x,\frac{T}{2}), & \text{if } t \in [0, \frac{T}{2}]. \end{cases} \quad (6.5.87)$$

We obtain from the above inequalities and from Carleman estimate

$$|u(0)|^2_{L^2(\Omega)} + \int_0^T\int_\Omega \tilde{\varphi}^3 e^{-2s\tilde{\eta}}|u|^2 dxdt \quad (6.5.88)$$
$$\leq C\int_0^T\int_\Omega e^{-2s\tilde{\eta}}|h|^2 dxdt + C\int_0^T\int_\omega \tilde{\varphi}^3 e^{-2s\tilde{\eta}}|u|^2 dxdt.$$

This gives the desired observability inequality (6.3.32) with the weight

$$\rho = \frac{e^{s\tilde{\eta}}}{\tilde{\varphi}^{\frac{3}{2}}}.$$

This completes the proof of Theorem 6.1.2 in the case we have a right-hand side g^0 satisfying

$$\frac{e^{s\tilde{\eta}}}{\tilde{\varphi}^{\frac{3}{2}}} g^0 \in L^2(0,T;L^2(\Omega)).$$

6.6. *Another Strategy*

We will present here the original strategy given by [11] which can be useful for applications to the controllability of nonlinear parabolic equations. With this strategy we will obtain exponentially decreasing controls and solutions for the null controllability problem.

Let us define

$$X_0 = \{w \in C^\infty(\overline{\Omega} \times [0.T]), \quad w = 0 \quad \text{on } \Gamma \times (0,T)\}. \tag{6.6.89}$$

On X_0 we define the bilinear form

$$a(w,\tilde{w}) = \int_{\Omega\times(0,T)} e^{-2s\tilde{\eta}}(-\frac{\partial w}{\partial t} + L^* w)(-\frac{\partial \tilde{w}}{\partial t} + L^*\tilde{w}) + \int_{\omega\times(0,T)} \tilde{\varphi}^3 e^{-2s\tilde{\eta}} w\tilde{w}. \tag{6.6.90}$$

Because of Carleman inequality and the observability inequality we have

$$\forall w \in X_0, \quad |w(0)|^2_{L^2(\Omega)} + \int_{\Omega\times(0,T)} \tilde{\varphi}^3 e^{-2s\tilde{\eta}} |w|^2 \le Ca(w,w).$$

Therefore $a(\cdot,\cdot)$ is a scalar product on X_0. Let us define X to be the completion of X_0 with respect to this scalar product. Then of course X is a Hilbert space for this scalar product and we have

$$\forall w \in X, \quad |w(0)|^2_{L^2(\Omega)} + \int_{\Omega\times(0,T)} \tilde{\varphi}^3 e^{-2s\tilde{\eta}} |w|^2 \le Ca(w,w). \tag{6.6.91}$$

Let us now define the linear form l on X by

$$\forall \tilde{w} \in X, \quad \langle l, \tilde{w}\rangle = (z^0, \tilde{w}(0)) + \int_{\Omega\times(0,T)} g^0 \tilde{w}. \tag{6.6.92}$$

Then if $z^0 \in L^2(\Omega)$ and $\frac{e^{s\tilde{\eta}}}{\tilde{\varphi}^{\frac{3}{2}}} g^0 \in L^2(0,T;L^2(\Omega))$, l is a continuous linear form on X. From Lax-Milgram Theorem, it follows that there exists a unique solution $w \in X$ to the following variational problem

$$a(w,\tilde{w}) = \langle l, \tilde{w}\rangle, \quad \forall \tilde{w} \in X. \tag{6.6.93}$$

Let us now call

$$z = e^{-2s\tilde{\eta}}(-\frac{\partial w}{\partial t} + L^* w), \tag{6.6.94}$$

$$h = -\tilde{\varphi}^3 e^{-2s\tilde{\eta}} w_{/\omega}. \tag{6.6.95}$$

Then we have, because $a(w,w) < +\infty$,

$$\int_{\Omega\times(0,T)} e^{2s\tilde{\eta}} |z|^2 < +\infty \tag{6.6.96}$$

and

$$\int_{\omega\times(0,T)} \frac{e^{2s\tilde{\eta}}}{\tilde{\varphi}^3}|h|^2 < +\infty. \qquad (6.6.97)$$

(Notice the + sign in the exponential weight.)

Moreover, (z,h) satisfy the following equation

$$\forall \tilde{w} \in X, \quad \int_{\Omega\times(0,T)} z(-\frac{\partial \tilde{w}}{\partial t} + L^*\tilde{w}) = (z^0, \tilde{w}(0)) + \int_{\Omega\times(0,T)} g^0\tilde{w} + \int_{\omega\times(0,T)} h\tilde{w}.$$

This can now be extended to all functions $\tilde{w}$ such that $(-\frac{\partial \tilde{w}}{\partial t} + L^*\tilde{w}) \in L^2(0,T;L^2(\Omega))$ such that $\tilde{w}(T) = 0$ and this shows that z is the (unique !) solution defined by transposition of problem (6.2.15). But we know that this problem has a solution in $C([0,T];L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$ so it must be the same solution.

Now we have (6.6.96) which implies

$$z(T) = 0$$

and which shows that the solution z decreases exponentially to 0 when $t \to T$. We also obtain from (6.6.97) an exponentially decreasing control h.

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