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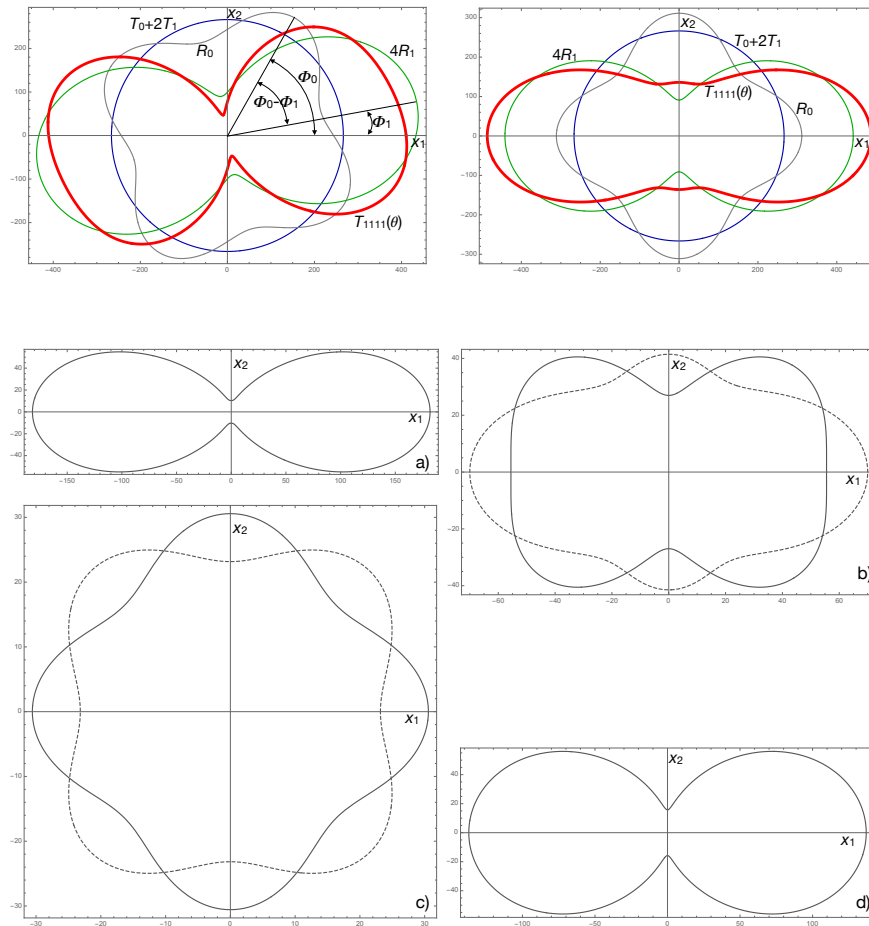
Paolo Vannucci¹

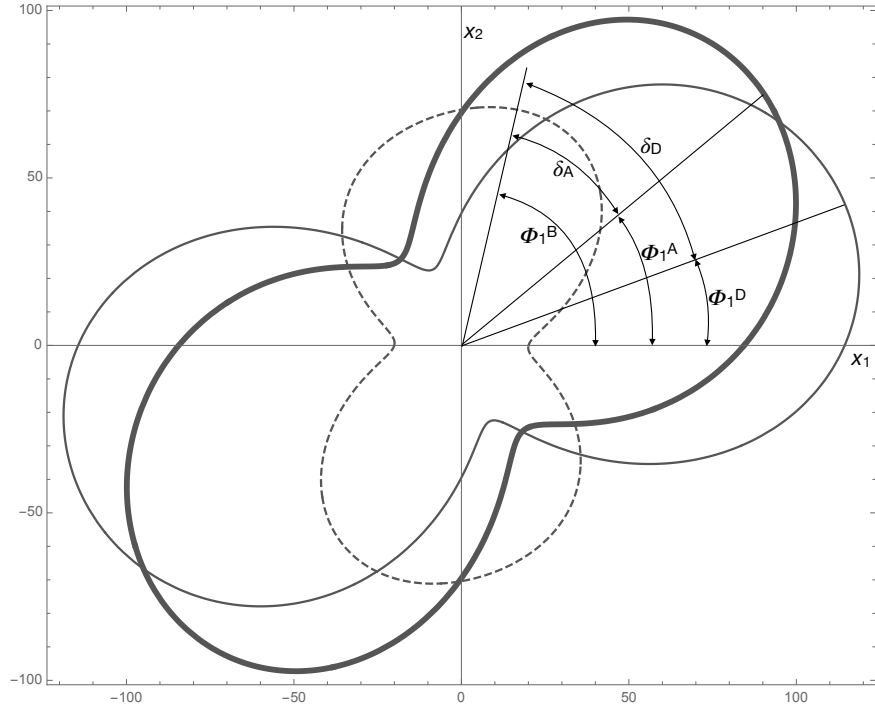
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September 30, 2023

Abstract

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On the bounds of the coupling tensor of anisotropic laminates

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September 29, 2023

Abstract

The problem of determining the elastic bounds of the coupling tensor for anisotropic laminates is addressed in this paper. It is shown how the invariant polar moduli of the coupling tensor interact with those of the extension and bending tensors to define all the conditions to be satisfied by such set of tensors describing the elastic behavior of an anisotropic laminate composed by identical layers. Some peculiar cases, interesting for applications, are also considered.

Key words: anisotropy; elastic moduli bounds; polar formalism; tensor invariants; laminates

PACS 46.25; 46.35; 62.20.de

MSC 74B05; 74E10; 74E30; 74K20

1 Introduction

In the theory of elastic laminates, the behavior of the plate is described by a law of the type

$$\begin{Bmatrix} \mathbf{N} \\ \mathbf{M} \end{Bmatrix} = \begin{bmatrix} h\mathbb{A} & \frac{h^2}{2}\mathbb{B} \\ \frac{h^2}{2}\mathbb{B} & \frac{h^3}{12}\mathbb{D} \end{bmatrix} \begin{Bmatrix} \boldsymbol{\varepsilon} \\ \boldsymbol{\kappa} \end{Bmatrix}, \quad (1)$$

where, [1,2], h is the plate's thickness, \mathbf{N} , \mathbf{M} are respectively the tensor of membrane forces and bending moments, $\boldsymbol{\varepsilon}$, $\boldsymbol{\kappa}$ the extension and curvature tensors, \mathbb{A} , \mathbb{D} the stiffness tensors of the extension and bending behaviors and \mathbb{B} the coupling tensor. \mathbb{A} , \mathbb{B} , \mathbb{D} are tensors of the elastic type, i.e. fourth-rank tensors with the major and minor symmetries but, unlike \mathbb{A} and \mathbb{D} , that are positive definite, \mathbb{B} is not defined. At least, this is what is commonly admitted, and not only for \mathbb{B} , but also for other coupling tensors describing some sort

of coupling between two physical phenomena, e.g. in the theory of quasi-crystals, [3]. Actually, the positive definiteness of \mathbb{A} and \mathbb{D} , like that of any other elastic tensor, is a consequence of the work, necessarily positive, done by the external forces. In the case of \mathbb{A} and \mathbb{D} , this is computed considering the laminate as uncoupled. The case of coupled laminates, i.e. having $\mathbb{B} \neq \mathbb{O}$, is almost unconsidered in the literature and \mathbb{B} is commonly seen as an *undefined* tensor, i.e. in general not positively nor negatively defined.

The positive definiteness of an elastic tensor gives as a result some bounds on the elastic moduli of the tensor in a given mathematical representation, e.g. the well known bounds on the Lamé's constants or on the Young's modulus and the Poisson's ratio for isotropic materials. More delicate is the case of anisotropic materials, [1, 4], for which a clear and definite set of bounds for the elastic moduli has not yet been defined in three-dimensional elasticity for any possible elastic syngony, namely for the most general one of triclinic materials. However, the same problem has been solved definitely in two-dimensional elasticity, using the polar formalism, [2, 5, 6]. By this mathematical technique, introduced as early as 1979 by G. Verchery, [7], any elastic tensor is represented by invariant moduli and angles. In this way, the bounds on the elastic tensor are given on its invariants, so intrinsically representing the elastic limits of the material. The same procedure has been used to determine the so-called *geometrical bounds*, [8], i.e. the bounds determining the elastic domain for a laminate when the stacking is considered.

The question of the bounds on \mathbb{B} remains open. This paper addresses exactly this subject and in particular it is a first attempt to give an answer to the following questions:

- i. is it possible to establish some bounds for the moduli of \mathbb{B} ?
- ii. how the known bounds on \mathbb{A} and \mathbb{D} are modified when $\mathbb{B} \neq \mathbb{O}$?
- iii. is it possible to give an explicit form to the bounds on the moduli of \mathbb{B} ?
- iv. can such bounds be given, in all the cases, in an invariant form?
- v. how the existence of some peculiar circumstances, e.g. a material symmetry, does affect the bounds on \mathbb{B} ?

This research has two motivations: the first, is a purely scientific question, interesting *per se*, to fill a gap still existing in the scientific literature: is it possible to give some bounds to tensor \mathbb{B} ? Another motivation can be found in optimization problems: any design problem for a coupled laminate is correctly formulated only if the design space is well determined, so as to properly define the feasibility domain.

The paper is organized as follows: in the next Section, the polar formalism is briefly recalled, especially for representing tensors \mathbb{A} , \mathbb{B} , \mathbb{D} . Then, the procedure used to determine the bounds on \mathbb{B} is detailed and subsequently the attempt to establish a general solution is explained. Some special cases are then considered and finally some conclusion is drawn in the end.

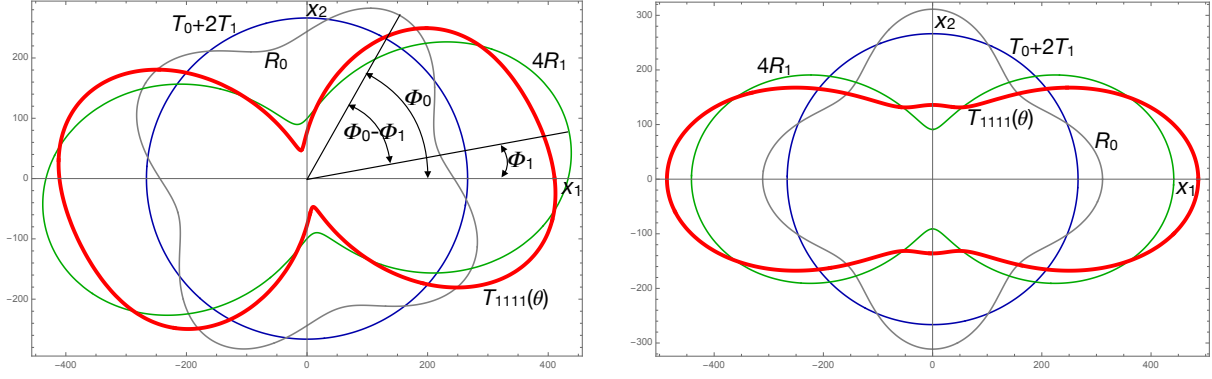


Figure 1: The decomposition of anisotropic plane elasticity in the polar method for a glass-epoxy layer with $T_0 = 92.38$ MPa, $T_1 = 86.97$ MPa, $R_0 = 44.86$ MPa, $R_1 = 43.82$ MPa, source [9]; left: a completely anisotropic material obtained if it was $\Phi_0 = \pi/3$, $\Phi_1 = \pi/20$, with, in blue (indicated also by $T_0 + 2T_1$), the isotropic phase, in gray (indicated by R_0) the R_0 phase, in green (indicated by $4R_1$) the R_1 phase and in thick red the overall result, i.e. the component $T_{1111}(\theta)$. Right: the true, orthotropic material, corresponding to $\Phi_0 = \Phi_1 = 0$.

2 Recall of the polar formalism for a laminate

For a given plane elastic tensor \mathbb{T} , the polar formalism allows to express the cartesian components at a direction θ as

$$\begin{aligned}
 T_{1111}(\theta) &= T_0 + 2T_1 + R_0 \cos 4(\Phi_0 - \theta) + 4R_1 \cos 2(\Phi_1 - \theta), \\
 T_{1112}(\theta) &= R_0 \sin 4(\Phi_0 - \theta) + 2R_1 \sin 2(\Phi_1 - \theta), \\
 T_{1122}(\theta) &= -T_0 + 2T_1 - R_0 \cos 4(\Phi_0 - \theta), \\
 T_{1212}(\theta) &= T_0 - R_0 \cos 4(\Phi_0 - \theta), \\
 T_{1222}(\theta) &= -R_0 \sin 4(\Phi_0 - \theta) + 2R_1 \sin 2(\Phi_1 - \theta), \\
 T_{2222}(\theta) &= T_0 + 2T_1 + R_0 \cos 4(\Phi_0 - \theta) - 4R_1 \cos 2(\Phi_1 - \theta).
 \end{aligned} \tag{2}$$

What is important to remark is the fact that the moduli T_0, T_1, R_0, R_1 as well as the difference of the angles $\Phi_0 - \Phi_1$ are tensor invariants; the value of one of the two polar angles, usually Φ_1 , fixes the frame. To be remarked that the polar method allows for a decomposition of anisotropic 2D elasticity into different *elastic phases*: an *isotropic phase*, characterized by the two invariants T_0 and T_1 , and two *anisotropic phases*, whose amplitudes are determined by the invariants R_0 and R_1 ; these two anisotropy phases are shifted of the angle $\Phi_0 - \Phi_1$, the fifth tensor invariant. To remark that any rotation of the angle θ is simply done in the polar method, it is sufficient to subtract θ from each one of the two polar angles (the complete demonstration of these fundamental result can be found in [2]). A general sketch of the decomposition of elasticity into elastic phases is given in Fig. 1, where the angles Φ_0, Φ_1 and $\Phi_0 - \Phi_1$ are also indicated.

Thanks to the polar formalism, the elastic symmetries are determined by the following values of the invariants:

- i. ordinary orthotropy: $\Phi_0 - \Phi_1 = k\frac{\pi}{4}$, $k \in \{0, 1\}$;
- ii. R_0 -orthotropy: $R_0 = 0$, [10];

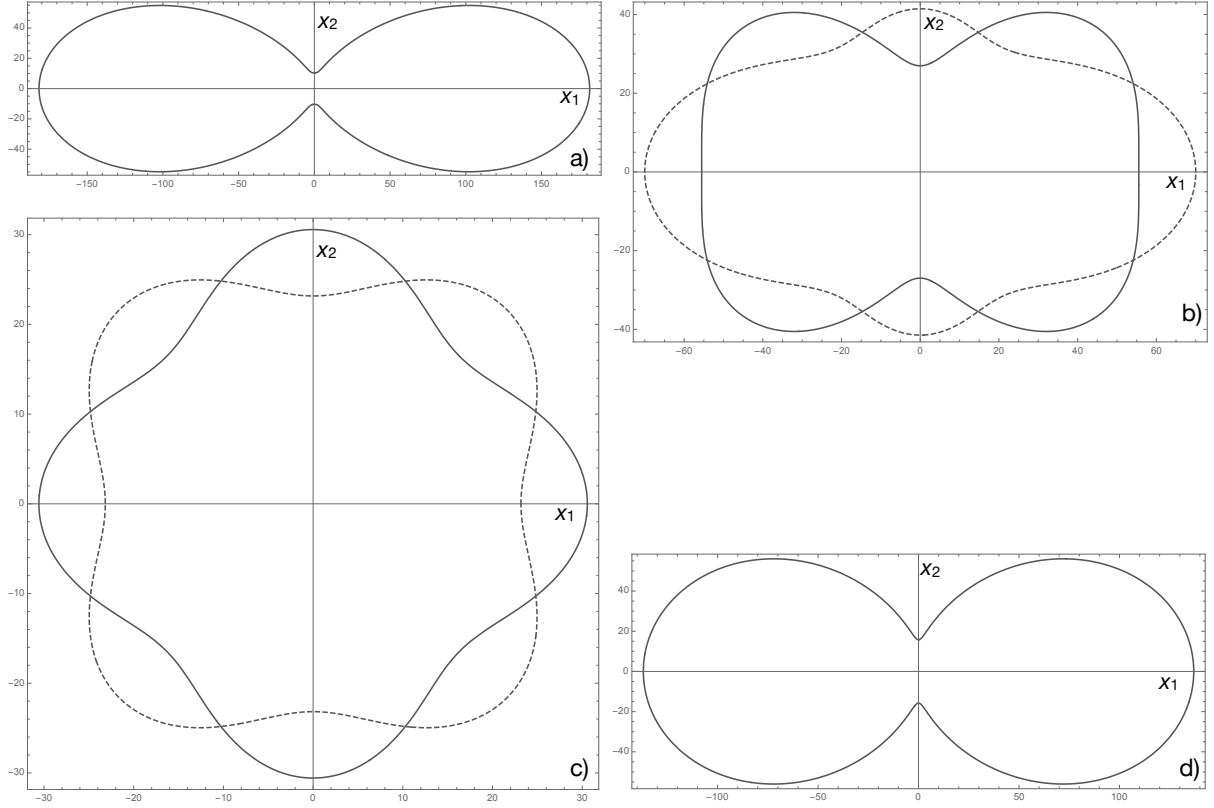


Figure 2: Polar diagrams of $T_{1111}(\theta)$ for some examples of elastic symmetries in 2D elasticity; a) a $k = 0$ orthotropic ply (T300/5208 carbon epoxy ply, [13]); b) a $k = 1$ orthotropic ply (braided carbon-epoxy BR45a ply, [14]); with a dashed line: a material with the same moduli but with $k = 0$; c) a square symmetric ply, $R_1 = 0$ (carbon epoxy balanced fabric, [15]); with a dashed line: the same material but with $\Phi_0 = \pi/4$; d) a R_0 -orthotropic material, obtained superposing two T300/5208 carbon epoxy plies rotated of $\pi/4$.

iii. square symmetry: $R_1 = 0$;

iv. isotropy: $R_0 = R_1 = 0$.

These elastic symmetries are the only possible ones in 2D elasticity; in particular, we see that two cases of ordinary orthotropy can exist, sharing the same values of the invariants T_0, T_1, R_0, R_1 . Moreover, square symmetry is the 2D corresponding of the 3D cubic symmetry: it is the case of a layer having two couples of mutually orthogonal symmetry axes, rotated of $\pi/4$, and with the same values of the elastic moduli along the two orthogonal axes. It is, namely, the case of layers reinforced by balanced fabrics, i.e. by fabrics having the same amount of fibers in warp and weft (that are often erroneously considered as isotropic), [11]. Finally, the case of R_0 -orthotropy has been discovered for the first time in 2D elasticity thanks to the polar method, [10], and later also in 3D elasticity, [12]. In Fig. 2, some examples of the above possible cases are given.

The above polar transformations apply also to tensors \mathbb{A}, \mathbb{B} and \mathbb{D} ; in particular, when a laminate is composed by identical layers, the case considered in this research, [16, 17], then (we indicate by a superscript A, B or D a polar quantity of \mathbb{A}, \mathbb{B} or \mathbb{D} respectively,

while a polar quantity of the basic layer has no superscript)

$$T_0^A = T_0^D = T_0, \quad T_1^A = T_1^D = T_1, \quad T_0^B = T_1^B = 0. \quad (3)$$

The polar formalism can be applied also to other situations, e.g. to the piezoelectric tensor, [18], or to thermo-elastic problems, [19, 20]. Some special cases of elastic non-classical materials have also been studied through the polar formalism, [21–23], however in this research we will consider only classical elastic tensors.

Regarding a second-rank symmetric tensor \mathbf{L} , in the polar formalism it is

$$\begin{aligned} L_{11}(\theta) &= T + R \cos 2(\Phi - \theta), \\ L_{12}(\theta) &= R \sin 2(\Phi - \theta), \\ L_{22}(\theta) &= T - R \cos 2(\Phi - \theta), \end{aligned} \quad (4)$$

with T, R two invariants and Φ an angle determined by the choice of the frame. About these relations, we can rephrase the comments given for eq. (2), adding that actually they correspond to the analytical expression of the well known graphical construction of the Mohr's circle, [24]. In the following, we will indicate by $t_\varepsilon, r_\varepsilon, \varphi_\varepsilon$ the polar components of ε and by $t_\kappa, r_\kappa, \varphi_\kappa$ those of κ .

3 Statement of the problem

For a coupled laminate, i.e. with $\mathbb{B} \neq \mathbb{O}$, the density of the elastic energy per unit of area of the plate is

$$U = \frac{1}{2} \begin{Bmatrix} \mathbf{N} \\ \mathbf{M} \end{Bmatrix} \cdot \begin{Bmatrix} \varepsilon \\ \kappa \end{Bmatrix} = \frac{1}{2} \begin{Bmatrix} \varepsilon \\ \kappa \end{Bmatrix} \cdot \begin{bmatrix} h\mathbb{A} & \frac{h^2}{2}\mathbb{B} \\ \frac{h^2}{2}\mathbb{B} & \frac{h^3}{12}\mathbb{D} \end{bmatrix} \begin{Bmatrix} \varepsilon \\ \kappa \end{Bmatrix}, \quad (5)$$

i.e., for being $\mathbb{B} = \mathbb{B}^\top$ by virtue of the major symmetries, [25],

$$U = \frac{h}{24} (12 \varepsilon \cdot \mathbb{A} \varepsilon + 12h \varepsilon \cdot \mathbb{B} \kappa + h^2 \kappa \cdot \mathbb{D} \kappa). \quad (6)$$

The energy density U must be positive for each possible strain state, i.e. $\forall \varepsilon, \kappa$; this is the condition leading to express the bounds for the elastic moduli. Following an approach first introduced by Verchery and detailed in [2, 26], we express all the tensors in the previous equation by their polar components, once fixed $\theta = 0$. Some standard passages lead to

$$\begin{aligned} U &= 2h \{ 2T_1 t_\varepsilon^2 + [T_0 + R_0^A \cos 4(\Phi_0^A - \varphi_\varepsilon)] r_\varepsilon^2 + 4R_1^A t_\varepsilon r_\varepsilon \cos 2(\Phi_1^A - \varphi_\varepsilon) \} \\ &\quad + 2h^2 \{ 2R_1^B t_\varepsilon r_\kappa \cos 2(\Phi_1^B - \varphi_\kappa) + 2R_1^B t_\kappa r_\varepsilon \cos 2(\Phi_1^B - \varphi_\varepsilon) \} \\ &\quad + R_0^B r_\varepsilon r_\kappa \cos 2(2\Phi_0^B - \varphi_\varepsilon - \varphi_\kappa) \} \\ &\quad + \frac{h^3}{6} \{ 2T_1 t_\kappa^2 + [T_0 + R_0^D \cos 4(\Phi_0^D - \varphi_\kappa)] r_\kappa^2 + 4R_1^D t_\kappa r_\kappa \cos 2(\Phi_1^D - \varphi_\kappa) \}. \end{aligned} \quad (7)$$

In this expression the first term in curly braces is due to extension, the second one to coupling and the third one to bending. Let us order the polar moduli of ε and κ as a

column vector $\{v\}$:

$$\{v\} = \begin{Bmatrix} t_\varepsilon \\ r_\varepsilon \\ t_\kappa \\ r_\kappa \end{Bmatrix}; \quad (8)$$

then, we can rewrite U as the quadratic form

$$U = \frac{h}{24} \{v\}^\top [M] \{v\}, \quad (9)$$

where $[M]$ is the 4×4 symmetric matrix

$$[M] = \begin{bmatrix} 96T_1 & 96R_1^A \cos 2(\Phi_1^A - \varphi_\varepsilon) & 0 & 48hR_1^B \cos 2(\Phi_1^B - \varphi_\kappa) \\ 96R_1^A \cos 2(\Phi_1^A - \varphi_\varepsilon) & 48[T_0 + R_0^A \cos 4(\Phi_0^A - \varphi_\varepsilon)] & 48hR_1^B \cos 2(\Phi_1^B - \varphi_\varepsilon) & 24hR_0^B \cos 2(2\Phi_0^B - \varphi_\varepsilon - \varphi_\kappa) \\ 0 & 48hR_1^B \cos 2(\Phi_1^B - \varphi_\varepsilon) & 8h^2T_1 & 8h^2R_1^D \cos 2(\Phi_1^D - \varphi_\kappa) \\ 48hR_1^B \cos 2(\Phi_1^B - \varphi_\kappa) & 24hR_0^B \cos 2(2\Phi_0^B - \varphi_\varepsilon - \varphi_\kappa) & 8h^2R_1^D \cos 2(\Phi_1^D - \varphi_\kappa) & 4h^2[T_0 + R_0^D \cos 4(\Phi_0^D - \varphi_\kappa)] \end{bmatrix}. \quad (10)$$

So, $U > 0 \forall \varepsilon, \kappa \iff [M]$ is positive definite $\forall \varphi_\varepsilon, \varphi_\kappa$. This happens if and only if, see [27] p. 340,

$$M1 = 96T_1 > 0, \quad (11)$$

$$M2 = \det \begin{bmatrix} 96T_1 & 96R_1^A \cos 2(\Phi_1^A - \varphi_\varepsilon) \\ 96R_1^A \cos 2(\Phi_1^A - \varphi_\varepsilon) & 48[T_0 + R_0^A \cos 4(\Phi_0^A - \varphi_\varepsilon)] \end{bmatrix} > 0 \forall \varphi_\varepsilon, \quad (12)$$

$$M3 = \det \begin{bmatrix} 96T_1 & 96R_1^A \cos 2(\Phi_1^A - \varphi_\varepsilon) & 0 \\ 96R_1^A \cos 2(\Phi_1^A - \varphi_\varepsilon) & 48[T_0 + R_0^A \cos 4(\Phi_0^A - \varphi_\varepsilon)] & 48hR_1^B \cos 2(\Phi_1^B - \varphi_\varepsilon) \\ 0 & 48hR_1^B \cos 2(\Phi_1^B - \varphi_\varepsilon) & 8h^2T_1 \end{bmatrix} > 0 \forall \varphi_\varepsilon, \quad (13)$$

$$M4 = \det[M] > 0 \forall \varphi_\varepsilon, \varphi_\kappa. \quad (14)$$

The first condition, eq. (11), is redundant: $T_1 > 0$ is a general condition for this polar modulus, see [2] p. 154. Because T_1 is a modulus of the basic layer, i.e. of a real material, this condition is automatically satisfied. The other conditions on $M2$, $M3$ and $M4$ are discussed in the next Section. Before doing that, we notice that only the condition on $M4$ concerns also the curvature field, while those on $M2$ and $M3$ depend just on the extension field.

4 General solution

We consider first the different conditions on $M2$, $M3$ and $M4$, then we examine them together in the next Section.

4.1 Condition on M2

Equation (12) gives the following condition for the positive definiteness of $[M]$:

$$T_1[T_0 + R_0^A \cos 4(\Phi_0^A - \varphi_\varepsilon)] - 2R_1^A{}^2 \cos^2 2(\Phi_1^A - \varphi_\varepsilon) > 0 \forall \varphi_\varepsilon. \quad (15)$$

This condition is exactly the same already found in [26] for proving the bounds on the polar moduli of an anisotropic layer; here, the condition refers to tensor \mathbb{A} . Following the

same steps outlined in [2] or in [26], it can be proved that eq. (15) is equivalent to the three conditions

$$\begin{aligned} T_0 - R_0^A &> 0, \\ T_0 T_1 - R_1^{A^2} &> 0, \\ T_1(T_0^2 - R_0^{A^2}) - 2R_1^{A^2}(T_0 - R_0^A \cos 4\Phi_A) &> 0, \end{aligned} \tag{16}$$

where $\Phi_A = \Phi_0^A - \Phi_1^A$, an invariant of \mathbb{A} . Actually, as shown in [2,26], the second condition above is less restrictive than the third one, so that it can be discarded. Finally, the condition on $M2$ becomes

$$\begin{aligned} T_0 - R_0^A &> 0, \\ T_1(T_0^2 - R_0^{A^2}) - 2R_1^{A^2}(T_0 - R_0^A \cos 4\Phi_A) &> 0. \end{aligned} \tag{17}$$

We remark that conditions (17) are written in terms of tensor invariants of \mathbb{A} , so they are frame independent.

Of course, the order in which the components of ε and of κ appear in vector $\{v\}$, eq. (8) is completely arbitrary, any other order being allowed. In particular, if one chooses the order putting first the components of κ , then those of ε , i.e. if we had put

$$\{v\} = \begin{Bmatrix} t_\kappa \\ r_\kappa \\ t_\varepsilon \\ r_\varepsilon \end{Bmatrix}, \tag{18}$$

then we had found for $M2$ a similar expression, but with the index D replacing the index A everywhere, i.e. the conditions would concern in this case the moduli of \mathbb{D} and not those of \mathbb{A} . Because of the arbitrariness of the order of the components of $\{v\}$, we can conclude that actually the above conditions (15) and (17) must hold necessarily also for the components of \mathbb{D} :

$$T_1[T_0 + R_0^D \cos 4(\Phi_0^D - \varphi_\kappa)] - 2R_1^{D^2} \cos^2 2(\Phi_1^D - \varphi_\kappa) > 0 \quad \forall \varphi_\kappa \tag{19}$$

and

$$\begin{aligned} T_0 - R_0^D &> 0, \\ T_1(T_0^2 - R_0^{D^2}) - 2R_1^{D^2}(T_0 - R_0^D \cos 4\Phi_D) &> 0, \end{aligned} \tag{20}$$

with of course $\Phi_D = \Phi_0^D - \Phi_1^D$.

4.2 Condition on M3

Developing the determinant in eq. (13), we get the condition

$$\begin{aligned} h^2 T_1 [T_0 T_1 + T_1 R_0^A \cos 4(\Phi_0^A - \varphi_\varepsilon) - 2R_1^{A^2} \cos^2 2(\Phi_1^A - \varphi_\varepsilon) \\ - 6R_1^{B^2} \cos^2 2(\Phi_1^B - \varphi_\varepsilon)] > 0 \quad \forall \varphi_\varepsilon. \end{aligned} \tag{21}$$

To solve this condition, we consider just the term in square brackets, as $T_1 > 0$, as already said; moreover, we introduce the angles

$$\alpha = \Phi_1^A - \varphi_\varepsilon \Rightarrow \Phi_0^A - \varphi_\varepsilon = \Phi_A + \alpha, \tag{22}$$

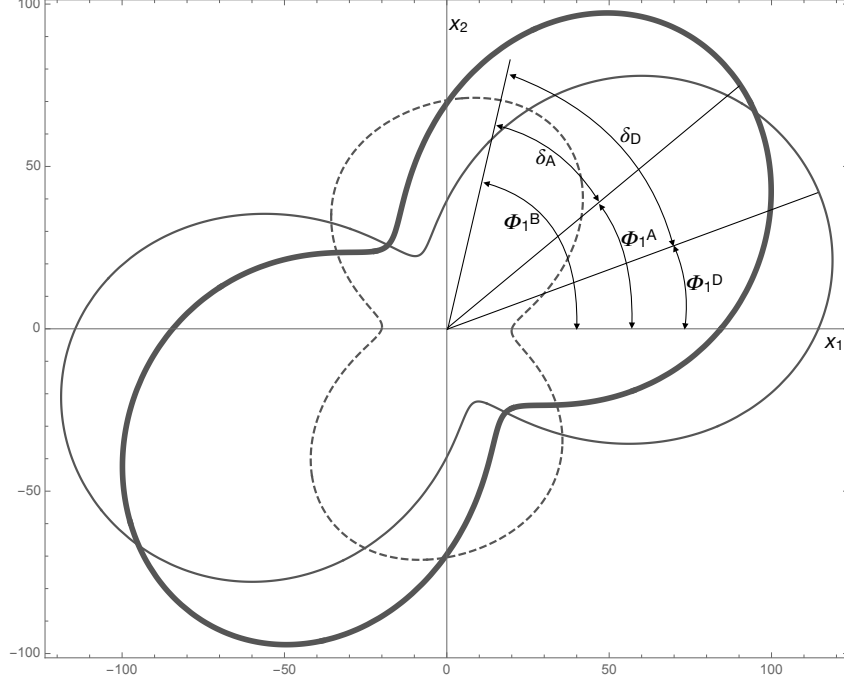


Figure 3: Polar diagrams of $A_{1111}(\theta)$, thick line, $B_{1111}(\theta)$, dashed line, and $D_{1111}(\theta)$, thin line, with indication of the polar angles $\Phi_1^A, \Phi_1^B, \Phi_1^D$ and of the shift angles δ_A and δ_D , for the general case of a completely anisotropic coupled laminate.

and

$$\delta_A = \Phi_1^B - \Phi_1^A \Rightarrow \Phi_1^B - \varphi_\varepsilon = \delta_A + \alpha. \quad (23)$$

The angle δ_A is the *shift angle* of tensor \mathbb{A} with respect to tensor \mathbb{B} , whose geometrical meaning is shown by Fig. 3. With this, eq. (21) is equivalent to

$$T_1[T_0 + R_0^A \cos 4(\Phi_A + \alpha)] > 2R_1^{A^2} \cos^2 2\alpha + 6R_1^{B^2} \cos^2 2(\delta_A + \alpha) > 0 \quad \forall \alpha. \quad (24)$$

Using standard trigonometry, this can be transformed first to

$$\begin{aligned} T_0 T_1 - R_1^{A^2} - 3R_1^{B^2} &> (T_1 R_0^A \sin 4\Phi_A + 3R_1^{B^2} \sin 4\delta_A) \sin 4\alpha - \\ &- (T_1 R_0^A \cos 4\Phi_A - 3R_1^{B^2} \cos 4\delta_A - R_1^{A^2}) \cos 4\alpha \quad \forall \alpha, \end{aligned} \quad (25)$$

then to

$$\begin{aligned} T_0 T_1 - R_1^{A^2} - 3R_1^{B^2} &> [(T_1 R_0^A \sin 4\Phi_A + 3R_1^{B^2} \sin 4\delta_A)^2 + \\ &+ (T_1 R_0^A \cos 4\Phi_A - 3R_1^{B^2} \cos 4\delta_A - R_1^{A^2})^2]^{\frac{1}{2}} \cos 4(\alpha + \omega) \quad \forall \alpha, \end{aligned} \quad (26)$$

with

$$\omega = \frac{1}{4} \arctan \frac{T_1 R_0^A \sin 4\Phi_A + 3R_1^{B^2} \sin 4\delta_A}{T_1 R_0^A \cos 4\Phi_A - 3R_1^{B^2} \cos 4\delta_A - R_1^{A^2}}. \quad (27)$$

For being satisfied for each possible value of α , eq. (26) gives the condition

$$\begin{aligned} T_0 T_1 - R_1^{A^2} - 3R_1^{B^2} &> [(T_1 R_0^A \sin 4\Phi_A + 3R_1^{B^2} \sin 4\delta_A)^2 + \\ &+ (T_1 R_0^A \cos 4\Phi_A - 3R_1^{B^2} \cos 4\delta_A - R_1^{A^2})^2]^{\frac{1}{2}}, \end{aligned} \quad (28)$$

solved into the two conditions

$$T_0 T_1 - R_1^{A^2} - 3R_1^{B^2} > 0 \quad (29)$$

and

$$\begin{aligned} (T_0 T_1 - R_1^{A^2} - 3R_1^{B^2})^2 &> (T_1 R_0^A \sin 4\Phi_A + 3R_1^{B^2} \sin 4\delta_A)^2 + \\ &+ (T_1 R_0^A \cos 4\Phi_A - 3R_1^{B^2} \cos 4\delta_A - R_1^{A^2})^2. \end{aligned} \quad (30)$$

This last can be easily transformed to

$$\begin{aligned} T_1^2 \left(T_0^2 - R_0^{A^2} \right) &+ 6T_1 R_0^A R_1^{B^2} \cos 4(\Phi_A + \delta_A) - \\ &- 2R_1^{A^2} \left[T_1 (T_0 - R_0^A \cos 4\Phi_A) - 3R_1^{B^2} \right] - \\ &- 6R_1^{B^2} \left(T_0 T_1 + R_1^{A^2} \cos 4\delta_A \right) > 0. \end{aligned} \quad (31)$$

Equations (29) and (31) are two bounds into which the condition on $M3$, eq. (13), is reduced. Both of them concern moduli of \mathbb{A} and \mathbb{B} , so these bounds describe the interactions existing between the moduli of these two tensors. Unlike eq. (29), depending exclusively upon tensor invariants, the condition in eq. (31) depends also on δ_A , which is not a tensor invariant. However, δ_A is a shift angle, so as such frame independent. Hence, also these bounds, corresponding to the condition $M3$, are frame independent like those deriving from the condition on $M2$.

Using once more the remark done in the previous Section about the order to give to the components of vector $\{v\}$ and operating like before, two more necessary bounds, concerning the moduli of \mathbb{B} and \mathbb{D} can be written as well, completely similar to eqs. (29) and (31):

$$\begin{aligned} T_0 T_1 - R_1^{D^2} - 3R_1^{B^2} &> 0, \\ T_1^2 \left(T_0^2 - R_0^{D^2} \right) &+ 6T_1 R_0^D R_1^{B^2} \cos 4(\Phi_D + \delta_D) - \\ &- 2R_1^{D^2} \left[T_1 (T_0 - R_0^D \cos 4\Phi_D) - 3R_1^{B^2} \right] - \\ &- 6R_1^{B^2} \left(T_0 T_1 + R_1^{D^2} \cos 4\delta_D \right) > 0, \end{aligned} \quad (32)$$

where δ_D is the *shift angle* between tensors \mathbb{B} and \mathbb{D} , see again Fig. 3:

$$\delta_D = \Phi_1^B - \Phi_1^D. \quad (33)$$

4.3 Condition on M4

The development of the determinant in eq. (14) leads to a condition much more cumbersome to be solved than the previous ones, for two reasons: on the one hand, the expression of this determinant is very complicate and, on the other hand, it must be positive for each possible value of φ_ε and of φ_κ , i.e. there are two possible independent fields to be taken into account simultaneously: strains and curvatures.

Developing the determinant in eq. (14) leads to the following condition

$$\begin{aligned}
& T_0^2 T_1^2 - 2R_1^A T_0 T_1 \cos^2 2(\Phi_1^A - \varphi_\varepsilon) + R_0^D T_0 T_1^2 \cos 4(\Phi_0^D - \varphi_\kappa) - \\
& - 2R_0^D R_1^A T_1 \cos^2 2(\Phi_1^A - \varphi_\varepsilon) \cos 4(\Phi_0^D - \varphi_\kappa) - \\
& - 6R_1^{B^2} T_0 T_1 \cos^2 2(\Phi_1^B - \varphi_\kappa) + 6R_1^{B^2} \cos^2 2(\Phi_1^B - \Phi_\varepsilon) \times \\
& \times \left[3R_1^{B^2} - T_0 T_1 - R_0^D T_1 \cos 4(\Phi_0^D - \varphi_\kappa) + 3R_1^{B^2} \cos 4(\Phi_1^B - \varphi_\kappa) \right] - \\
& - R_0^A T_1 \cos 4(\Phi_0^A - \varphi_\varepsilon) \left[3R_1^{B^2} + R_1^{D^2} - T_0 T_1 - R_0^D T_1 \cos 4(\Phi_0^D - \varphi_\kappa) + \right. \\
& \left. + 3R_1^{B^2} \cos 4(\Phi_1^B - \varphi_\kappa) + R_1^{D^2} \cos 4(\Phi_1^D - \varphi_\kappa) \right] + \\
& + 12R_0^B R_1^A R_1^B T_1 \cos 2(\Phi_1^A - \varphi_\varepsilon) \cos 2(\Phi_1^B - \varphi_\kappa) \cos 2(2\Phi_0^B - \varphi_\varepsilon - \varphi_\kappa) - \\
& - 2R_1^{D^2} T_0 T_1 \cos^2 2(\Phi_1^D - \varphi_\kappa) - 3R_0^{B^2} T_1^2 \cos^2 2(2\Phi_0^B - \varphi_\varepsilon - \varphi_\kappa) + \\
& + 4R_1^{A^2} R_1^{D^2} \cos^2 2(\Phi_1^A - \varphi_\varepsilon) \cos^2 2(\Phi_1^D - \varphi_\kappa) + \\
& + 12R_1^B R_1^D \cos 2(\Phi_1^B - \varphi_\varepsilon) \cos 2(\Phi_1^D - \varphi_\kappa) \left[R_0^B T_1 \cos 2(2\Phi_0^B - \varphi_\varepsilon - \varphi_\kappa) - \right. \\
& \left. - 2R_1^A R_1^B \cos 2(\Phi_1^A - \varphi_\varepsilon) \cos 2(\Phi_1^B - \varphi_\kappa) \right] > 0 \quad \forall \varphi_\varepsilon, \varphi_\kappa;
\end{aligned} \tag{34}$$

this one can be further developed through extensive use of trigonometric formulae and introducing the angles Φ_A, Φ_D, δ_A and δ_D previously defined, along with the angle $\Phi_B = \Phi_0^B - \Phi_1^B$, an invariant of \mathbb{B} , and writing the condition for $\theta = \Phi_1^B$ instead than for $\theta = 0$, like done until now (to do this, it is sufficient, in the polar formalism, to subtract the angle Φ_1^B from all the polar angles, see [2], p. 146). The result is

$$\begin{aligned}
& \left\{ \left[T_0 T_1 + T_1 R_0^A \cos 4(\Phi_A - \delta_A - \varphi_\varepsilon) - 2R_1^{A^2} \cos^2 2(\delta_A + \varphi_\varepsilon) \right] \times \right. \\
& \times \left. \left[T_0 T_1 + T_1 R_0^D \cos 4(\Phi_D - \delta_D - \varphi_\kappa) - 2R_1^{D^2} \cos^2 2(\delta_D + \varphi_\kappa) \right] \right\} + \\
& + 36R_1^{B^4} \cos^2 2\varphi_\varepsilon \cos^2 2\varphi_\kappa - 6T_0 T_1 R_1^{B^2} (\cos^2 2\varphi_\varepsilon + \cos^2 2\varphi_\kappa) - \\
& - 3T_1^2 R_0^{B^2} \cos^2 2(2\Phi_B - \varphi_\varepsilon - \varphi_\kappa) - \\
& - 24R_1^A R_1^{B^2} R_1^D \cos 2(\delta_A + \varphi_\varepsilon) \cos 2\varphi_\varepsilon \cos 2(\delta_D + \varphi_\kappa) \cos 2\varphi_\kappa - \\
& - 6T_1 R_1^{B^2} \left[R_0^A \cos 4(\Phi_A - \delta_A - \varphi_\varepsilon) \cos^2 2\varphi_\kappa + \right. \\
& \left. + R_0^D \cos 4(\Phi_D - \delta_D - \varphi_\kappa) \cos^2 2\varphi_\varepsilon \right] + 12T_1 R_0^B R_1^B \cos 2(2\Phi_B - \varphi_\varepsilon - \varphi_\kappa) \times \\
& \times \left[R_1^A \cos 2(\delta_A + \varphi_\varepsilon) \cos 2\varphi_\kappa + R_1^D \cos 2(\delta_D + \varphi_\kappa) \cos 2\varphi_\varepsilon \right] > 0 \quad \forall \varphi_\varepsilon, \varphi_\kappa.
\end{aligned} \tag{35}$$

This form is particularly interesting because the term in curly braces is actually the product of the condition $M2$ for \mathbb{A} times the same condition for \mathbb{D} , cf. eqs. (15) and (19). The rest of the above expression depends on the invariants of \mathbb{B} , i.e. on R_0^B, R_1^B and Φ_B ; this part represents hence the influence of coupling on the elastic bounds of \mathbb{A} and \mathbb{D} . Whenever $\mathbb{B} = \mathbb{O}$, all this part vanishes and condition $M4$ becomes redundant, as obvious, because it reduces simply to conditions (17) and (20), to be satisfied simultaneously. We remark also that, like for the case of $M3$, eq. (35) depends only on tensor invariants and on the two shift angles δ_A and δ_D , so it is frame independent too.

Unfortunately, it is not possible to derive from the condition (35) any explicit bound for the elastic moduli. In fact, on the one hand a procedure like the one described in the previous section for $M3$ is no longer possible in this case and, on the other hand, the problem

could be tackled looking for the minimum of the expression in eq. (35), i.e. calculating the gradient with respect to φ_ε and φ_κ , then solving the equation obtained equating it to zero, searching the minimum among the solutions and imposing such a minimum to be positive. Trying to do this leads to some trigonometric equations that cannot be solved analytically, though this remains possible numerically, for any application case.

5 Special cases

Besides the conditions found above, the anisotropic moduli of \mathbb{A} , \mathbb{B} and \mathbb{D} , i.e. $R_0^A, R_0^B, R_0^D, R_1^A, R_1^B, R_1^D$ must be put non negative, as each one of them is the modulus of a complex quantity, [2], p. 144. Actually, T_0 and T_1 are also positive quantities, but no conditions are to be written for them because they are two elastic invariant moduli of the basic layer composing the laminate, so as moduli of a real material, they automatically satisfy the elastic bounds.

All the previous conditions resume to the set of inequalities

$$\begin{aligned}
R_0^A &\geq 0, \\
R_1^A &\geq 0, \\
R_0^B &\geq 0, \\
R_1^B &\geq 0, \\
R_0^D &\geq 0, \\
R_1^D &\geq 0, \\
T_0 - R_0^A &> 0, \\
T_1(T_0^2 - R_0^{A^2}) - 2R_1^{A^2}(T_0 - R_0^A \cos 4\Phi_A) &> 0, \\
T_0T_1 - R_1^{A^2} - 3R_1^{B^2} &> 0, \\
T_1^2 (T_0^2 - R_0^{A^2}) + 6T_1R_0^AR_1^{B^2} \cos 4(\Phi_A + \delta_A) - \\
- 2R_1^{A^2} [T_1 (T_0 - R_0^A \cos 4\Phi_A) - 3R_1^{B^2}] - \\
- 6R_1^{B^2} (T_0T_1 + R_1^{A^2} \cos 4\delta_A) &> 0 \\
\min[M4] &> 0.
\end{aligned} \tag{36}$$

In fact, eqs. (20) and (32) are redundant for establishing the positive definiteness of matrix $[M]$. Though the last of the previous conditions cannot be given in an analytical explicit form, something more can be said in some particular cases, detailed below.

We notice also that the condition on $M1$ concerns just the material, and it is automatically satisfied, that on $M2$ concerns exclusively \mathbb{A} (or alternatively \mathbb{D}), the one on $M3$ regards \mathbb{A} (or \mathbb{D}) and \mathbb{B} together and finally the condition on $M4$ concerns all the three tensors, \mathbb{A} , \mathbb{B} and \mathbb{D} together. Finally, it is immediate to check that whenever $\mathbb{B} = \mathbb{O}$ the conditions

reduce simply to those already known, for \mathbb{A} and for \mathbb{D} separately:

$$\begin{aligned}
R_0^A &\geq 0, \\
R_1^A &\geq 0, \\
T_0 - R_0^A &> 0, \\
T_1(T_0^2 - R_0^{A2}) - 2R_1^{A2}(T_0 - R_0^A \cos 4\Phi_A) &> 0, \\
R_0^D &\geq 0, \\
R_1^D &\geq 0, \\
T_0 - R_0^D &> 0, \\
T_1(T_0^2 - R_0^{D2}) - 2R_1^{D2}(T_0 - R_0^D \cos 4\Phi_D) &> 0.
\end{aligned} \tag{37}$$

This actually means that not only the matrix in eq. (1) or, which is equivalent, matrix $[M]$, eq. (10), must be positive definite, but also \mathbb{A} and \mathbb{D} separately, as well known for uncoupled laminates, but not \mathbb{B} . The bounds involving moduli of \mathbb{B} are not found imposing its positive definiteness: \mathbb{B} is actually not definite.

5.1 Aligned orthotropic tensors

Let us consider the case of *aligned orthotropic tensors*, i.e.

$$\begin{aligned}
\delta_A = \lambda_A \frac{\pi}{2}, \quad \delta_D = \lambda_D \frac{\pi}{2}, \quad \Phi_A = k_A \frac{\pi}{4}, \quad \Phi_b = k_B \frac{\pi}{4}, \quad \Phi_D = k_D \frac{\pi}{4}, \\
\lambda_A, \lambda_D, k_A, k_B, k_D \in \{0, 1\}.
\end{aligned} \tag{38}$$

In such a case, very common in practice, it is easy to check that

$$\nabla M4 = 0 \iff \varphi_\varepsilon = h_\varepsilon \frac{\pi}{4}, \quad \varphi_\kappa = h_\kappa \frac{\pi}{4}, \quad h_\varepsilon, h_\kappa \in \{0, 1\}, \tag{39}$$

i.e. the minimum of $M4$ can be only in one of the four points

$$P_1 = (0, 0), \quad P_2 = \left(\frac{\pi}{4}, 0\right), \quad P_3 = \left(0, \frac{\pi}{4}\right), \quad P_4 = \left(\frac{\pi}{4}, \frac{\pi}{4}\right). \tag{40}$$

Calling, for the sake of conciseness, $M4_1, M4_2, M4_3$ and $M4_4$ the value taken by (35) at P_1, P_2, P_3 and P_4 respectively, then the condition on $M4$ can be put in the form

$$\min\{M4_1, M4_2, M4_3, M4_4\} > 0, \tag{41}$$

where

$$\begin{aligned}
M4_1 &= \left[T_0 T_1 + (-1)^{k_A} T_1 R_0^A - 2R_1^{A2}\right] \left[T_0 T_1 + (-1)^{k_D} T_1 R_0^D - 2R_1^{D2}\right] + \\
&\quad + 36R_1^{B4} - 3T_1^2 R_0^{B2} - 6T_1 R_1^{B2} \left[2T_0 + (-1)^{k_A} R_0^A + (-1)^{k_D} R_0^D\right] + \\
&\quad + 12(-1)^{k_B} T_1 R_0^B R_1^B \left[(-1)^{\lambda_A} R_1^A + (-1)^{\lambda_D} R_1^D\right] - \\
&\quad - 24(-1)^{\lambda_A} (-1)^{\lambda_D} R_1^A R_1^{B2} R_1^D, \\
M4_2 &= T_1 \left[T_0 - (-1)^{k_A} R_0^A\right] \left[T_1 (T_0 + (-1)^{k_D} R_0^D) - 2R_1^{D2} - 6R_1^{B2}\right], \\
M4_3 &= T_1 \left[T_0 - (-1)^{k_D} R_0^D\right] \left[T_1 (T_0 + (-1)^{k_A} R_0^A) - 2R_1^{A2} - 6R_1^{B2}\right], \\
M4_4 &= T_1^2 \left\{ \left[T_0 - (-1)^{k_A} R_0^A\right] \left[T_0 - (-1)^{k_D} R_0^D\right] - 3R_0^{B2} \right\}.
\end{aligned} \tag{42}$$

In such a situation, the conditions (17) on $M2$ become

$$\begin{aligned} T_0 - R_0^A &> 0, \\ [T_0 - (-1)^{k_A} R_0^A] [T_1 (T_0 + (-1)^{k_A} R_0^A) - 2R_1^{A^2}] &> 0, \end{aligned} \quad (43)$$

while those on $M3$, eqs. (29) and (30),

$$\begin{aligned} T_0 T_1 - R_1^{A^2} - 3R_1^{B^2} &> 0, \\ T_1 [T_0 - (-1)^{k_A} R_0^A] [T_1 (T_0 + (-1)^{k_A} R_0^A) - 2R_1^{A^2} - 6R_1^{B^2}] &> 0. \end{aligned} \quad (44)$$

Because $T_1 > 0$ and by eqs. (43)₁ and (44)₂, condition (43)₂ is redundant and the above conditions can be rewritten as

$$\begin{aligned} T_0 - R_0^A &> 0, \\ T_0 T_1 - R_1^{A^2} - 3R_1^{B^2} &> 0, \\ T_1 [T_0 + (-1)^{k_A} R_0^A] - 2R_1^{A^2} - 6R_1^{B^2} &> 0. \end{aligned} \quad (45)$$

A scrutiny of $M4_2$ and of $M4_3$ allows to see that if conditions (45)_{1,3} are considered in eqs. (42)_{2,3}, this will lead to two conditions that are exactly the analogous of those that could be obtained by eqs.(20) and (32), i.e. writing the conditions $M2$ and $M3$ also for \mathbb{D} .

Finally the whole set of conditions for the case at hand will be

$$\begin{aligned} R_0^A &\geq 0, \\ R_1^A &\geq 0, \\ R_0^B &\geq 0, \\ R_1^B &\geq 0, \\ R_0^D &\geq 0, \\ R_1^D &\geq 0, \\ T_0 - R_0^A &> 0, \\ T_0 T_1 - R_1^{A^2} - 3R_1^{B^2} &> 0, \\ T_1 [T_0 + (-1)^{k_A} R_0^A] - 2R_1^{A^2} - 6R_1^{B^2} &> 0, \\ \min\{M4_1, M4_2, M4_3, M4_4\} &> 0. \end{aligned} \quad (46)$$

It is likely that the above conditions constitute general bounds, i.e. that they are valid also for other more general situations, or, in other words, that the aligned orthotropic case is the worst one. However, this is just a conjecture that cannot be proved.

5.2 Square symmetric \mathbb{B}

Let us now consider the case of a square-symmetric coupling \mathbb{B} , i.e. a laminate designed to have $R_1^B = 0$. In such a case, eq. (36)₉ reduces to

$$T_0 T_1 - R_1^{A^2} > 0, \quad (47)$$

which is redundant, as previously discussed in Sect. 4.1. Moreover, eq. (36)₁₀ becomes exactly eq. (36)₈, so it can be discarded too. Actually, in this special case it is the whole set of conditions on $M3$ that it is redundant. This is quite natural, because when $R_1^B = 0$ the only parameter accounting for coupling disappears from $M3$, which reduces by consequence to conditions already accounted for by $M2$. Moreover, eq. (35) reduces to

$$\begin{aligned} & \left[T_0 T_1 + T_1 R_0^A \cos 4(\Phi_A - \delta_A - \varphi_\varepsilon) - 2R_1^{A^2} \cos^2 2(\delta_A + \varphi_\varepsilon) \right] \times \\ & \times \left[T_0 T_1 + T_1 R_0^D \cos 4(\Phi_D - \delta_D - \varphi_\kappa) - 2R_1^{D^2} \cos^2 2(\delta_D + \varphi_\kappa) \right] - \\ & - 3T_1^2 R_0^{B^2} \cos^2 2(2\Phi_B - \varphi_\varepsilon - \varphi_\kappa) > 0 \quad \forall \varphi_\varepsilon, \varphi_\kappa. \end{aligned} \quad (48)$$

Also in this case it is interesting to consider the case of orthotropic aligned tensors, eq. (38), putting conventionally $\Phi_1^B = \Phi_0^B$ (if $R_1^B = 0$, Φ_1^B is not defined). Repeating *verbatim* the steps already done in Sect. 5.1 gives eventually the conditions

$$\begin{aligned} R_0^A & \geq 0, \\ R_1^A & \geq 0, \\ R_0^B & \geq 0, \\ R_0^D & \geq 0, \\ R_1^D & \geq 0, \\ T_0 - R_0^A & > 0, \\ T_1 [T_0 + (-1)^{k_A} R_0^A] - 2R_1^{A^2} & > 0, \\ \min\{M4_1, M4_2, M4_3, M4_4\} & > 0. \end{aligned} \quad (49)$$

with now

$$\begin{aligned} M4_1 & = \left[T_1 (T_0 + (-1)^{k_A} R_0^A) - 2R_1^{A^2} \right] \times \\ & \times \left[T_1 (T_0 + (-1)^{k_D} R_0^D) - 2R_1^{D^2} \right] - 3T_1^2 R_0^{B^2}, \\ M4_2 & = T_1 [T_0 - (-1)^{k_A} R_0^A] \left[T_1 (T_0 + (-1)^{k_D} R_0^D) - 2R_1^{D^2} \right], \\ M4_3 & = T_1 [T_0 - (-1)^{k_D} R_0^D] \left[T_1 (T_0 + (-1)^{k_A} R_0^A) - 2R_1^{A^2} \right], \\ M4_4 & = T_1^2 \left\{ [T_0 - (-1)^{k_A} R_0^A] [T_0 - (-1)^{k_D} R_0^D] - 3R_0^{B^2} \right\}. \end{aligned} \quad (50)$$

5.3 Fully square symmetric laminates

A more particular case, and even more interesting for applications, is that of *fully square symmetric laminates*, i.e. of

$$R_1^A = R_1^B = R_1^D = 0, \quad (51)$$

which is, e.g., automatically get if $R_1 = 0$, [11], i.e. if the basic layer is square symmetric itself, like actually it is for the very common case of layers reinforced by balanced fabrics.

In such a case, the previous conditions (49) reduce to only

$$\begin{aligned}
R_0^A &\geq 0, \\
R_0^B &\geq 0, \\
R_0^D &\geq 0, \\
T_0 - R_0^A &> 0, \\
\min\{M4_1, M4_2, M4_3, M4_4\} &> 0.
\end{aligned} \tag{52}$$

with

$$\begin{aligned}
M4_1 &= [T_0 + (-1)^{k_A} R_0^A] [T_0 + (-1)^{k_D} R_0^D] - 3R_0^{B^2}, \\
M4_2 &= [T_0 - (-1)^{k_A} R_0^A] [T_0 + (-1)^{k_D} R_0^D], \\
M4_3 &= [T_0 - (-1)^{k_D} R_0^D] [T_0 + (-1)^{k_A} R_0^A], \\
M4_4 &= [T_0 - (-1)^{k_A} R_0^A] [T_0 - (-1)^{k_D} R_0^D] - 3R_0^{B^2}.
\end{aligned} \tag{53}$$

This special case of coupled laminates is interesting for applications because condition (51) is sufficient to obtain coupled thermally stable laminates, [20], i.e. coupled laminates that preserve their form also under a temperature change, like the one occurred during the curing phase of pre-preg layers.

5.4 R_0 -orthotropic laminates

Let us consider now the case of a laminate designed to have

$$R_0^A = R_0^B = R_0^D = 0, \tag{54}$$

which is a special case of orthotropy, named *R_0 -orthotropy*, [10,22]. This kind of laminates can be obtained simply stacking layers with $R_0 = 0$. For this special case, eqs. (36) reduce to (we recall that like T_1 , also $T_0 > 0$ automatically, because it is the modulus of a real material)

$$\begin{aligned}
R_1^A &\geq 0, \\
R_1^B &\geq 0, \\
R_1^D &\geq 0, \\
T_0 T_1 - 2R_1^{A^2} &> 0, \\
T_0 T_1 - R_1^{A^2} - 3R_1^{B^2} &> 0, \\
T_0^2 T_1^2 - 2R_1^{A^2} (T_0 T_1 - 3R_1^{B^2}) - 6R_1^{B^2} (T_0 T_1 + R_1^{A^2} \cos 4\delta_A) &> 0, \\
\min \left\{ \left[T_0 T_1 - 2R_1^{A^2} \cos^2 2(\delta_A + \varphi_\varepsilon) \right] \left[T_0 T_1 - 2R_1^{D^2} \cos^2 2(\delta_D + \varphi_\kappa) \right] + \right. \\
&+ 36R_1^{B^4} \cos^2 2\varphi_\varepsilon \cos^2 2\varphi_\kappa - 6T_0 T_1 R_1^{B^2} (\cos^2 2\varphi_\varepsilon + \cos^2 2\varphi_\kappa) - \\
&\left. - 24R_1^A R_1^{B^2} R_1^D \cos 2(\delta_A + \varphi_\varepsilon) \cos 2\varphi_\varepsilon \cos 2(\delta_D + \varphi_\kappa) \cos 2\varphi_\kappa \right\} &> 0.
\end{aligned} \tag{55}$$

If once again we consider the case of aligned tensors, eq. (38), then proceeding like in Sect. 5.1, we get that condition (55)₄ is redundant and finally

$$\begin{aligned}
R_1^A &\geq 0, \\
R_1^B &\geq 0, \\
R_1^D &\geq 0, \\
T_0T_1 - R_1^{A^2} - 3R_1^{B^2} &> 0, \\
T_0T_1 - 2R_1^{A^2} - 6R_1^{B^2} &> 0, \\
\min[M4_1, M4_2, M4_3] &> 0,
\end{aligned} \tag{56}$$

with now

$$\begin{aligned}
M4_1 &= \left(T_0T_1 - 2R_1^{A^2}\right) \left(T_0T_1 - 2R_1^{D^2}\right) - \\
&\quad - 12R_1^{B^2} \left[T_0T_1 - 3R_1^{B^2} + 2(-1)^{\lambda_A}(-1)^{\lambda_D} R_1^A R_1^D\right], \\
M4_2 &= T_0T_1 \left[T_0T_1 - 2R_1^{D^2} - 6R_1^{B^2}\right], \\
M4_3 &= T_0T_1 \left[T_0T_1 - 2R_1^{A^2} - 6R_1^{B^2}\right].
\end{aligned} \tag{57}$$

To notice that in this case $M4_4 = T_0^2T_1^2 > 0$ always.

5.5 Coupled isotropic laminates

A last particular and rather interesting case is that of coupled laminates having an isotropic behavior in extension and in bending. Such a kind of laminates can be obtained in different ways, e.g. applying the Werren and Norris conditions, [28] to laminates of the quasi-trivial type with $\mathbb{A} = \mathbb{D}$, [16, 29]; an example of this kind of plates is the 18-layers laminate whose stacking sequence is

$$[0^\circ, 60^\circ, -60_2^\circ, 60_2^\circ, -60^\circ, 0^\circ, 60_2^\circ, 0^\circ, -60^\circ, 0^\circ, -60^\circ, 0_2^\circ, -60^\circ, 60^\circ].$$

In such a situation, it is $R_0^A = R_1^A = R_0^D = R_1^D = 0$, while all the angles $\Phi_A, \Phi_D, \delta_A, \delta_D$ are no longer defined. To notice that it is necessarily $\mathbb{A} = \mathbb{D}$, because they are reduced to the only isotropic part, i.e. to T_0 and T_1 , that are those of the basic layer, while \mathbb{B} is necessarily anisotropic, because $T_0^B = T_1^B = 0, R_0^B + R_1^B \neq 0$.

In this situation, eq. (35) becomes

$$\begin{aligned}
T_0^2T_1^2 + 36R_1^{B^4} \cos^2 2\varphi_\varepsilon \cos^2 2\varphi_\kappa - 6T_0T_1R_1^{B^2}(\cos^2 2\varphi_\varepsilon + \cos^2 2\varphi_\kappa) - \\
- 3T_1^2R_0^{B^2} \cos^2 2(2\Phi_B - \varphi_\varepsilon - \varphi_\kappa) > 0 \quad \forall \varphi_\varepsilon, \varphi_\kappa,
\end{aligned} \tag{58}$$

eq. (36)₉ is redundant with respect to eq. (36)₁₀ and finally conditions (36) reduce to only

$$\begin{aligned}
R_0^B &\geq 0, \\
R_1^B &\geq 0, \\
T_0T_1 - 6R_1^{B^2} &> 0, \\
\min[M4] &> 0.
\end{aligned} \tag{59}$$

Once more, if \mathbb{B} is orthotropic, i.e. $\Phi_B = k_B \frac{\pi}{4}$, $k_B \in \{0, 1\}$, then

$$\begin{aligned} M4_1 &= T_1^2(T_0^2 - 3R_0^{B^2}) - 12R_1^{B^2}(T_0T_1 - 3R_1^{B^2}), \\ M4_2 &= M4_3 = T_0T_1(T_0T_1 - 6R_1^{B^2}), \\ M4_4 &= T_1^2(T_0^2 - 3R_0^{B^2}), \end{aligned} \tag{60}$$

and conditions (59) become simply

$$\begin{aligned} R_0^B &\geq 0, \\ R_1^B &\geq 0, \\ T_0T_1 - 6R_1^{B^2} &> 0, \\ \min\{M4_1, M4_2, M4_4\} &> 0. \end{aligned} \tag{61}$$

6 Conclusion

Coming back to the questions posed in Sect. 1, the result found above show that:

- i. it is possible to establish some bounds involving *also* the moduli of \mathbb{B} but not *exclusively* these ones;
- ii. the moduli of \mathbb{A} and \mathbb{D} known for the case of uncoupled laminates are still valid but in addition some other bounds, relating the moduli of \mathbb{A} , \mathbb{B} and \mathbb{D} are added in the case of coupled laminates;
- iii. not all the bounds relating the moduli of \mathbb{A} , \mathbb{B} and \mathbb{D} can be given in an explicit form;
- iv. all the bounds depend exclusively on tensor invariants and shift angles, so all of them are frame independent;
- v. the existence of some kind of symmetries affects the bounds and normally allows to express them in a simpler form.

The previous results show also that \mathbb{B} remains an undefined tensor. The bounds found in this paper reveal hence the existence of some supplementary conditions to be satisfied by the moduli of \mathbb{A} and \mathbb{D} , these conditions imposing some sort of interaction with the moduli of \mathbb{B} . This result, unknown until now, is interesting *per se*, because it shows that coupling is not unbounded, but also in practical applications, namely in optimization problems, to correctly define the feasibility domain in the search of optimal coupled laminates.

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